A TREATISE ON

SPHERICAL TRIGONOMETRY

WITH APPLICATIONS TO

SPHERICAL GEOMETRY

AND NUMEROUS EXAMPLES

PART I.

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PART I.

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PREFACE

TO THE FIRST PART.

THE object of the present Treatise is to bring Spherical Trigonometry to the standard required for University Examinations, and demanded by the impulse given to mathematical subjects by modern text-books.

Simplicity of treatment has been constantly kept in view.

Part I. treats of the subject as far as the solution of Triangles, inclusive. In the text will be found all the Propositions usually contained in treatises on the subject, besides such other Theorems as appeared to us to be of special importance on account of their utility.

The Volume is replete with examples (in many cases worked out), the arrangement of which has been the subject of our special attention: our aim throughout being to place them in immediate connexion with the subject-matter of which they are illustrative.

At the end of each Chapter, Miscellaneous Examples bearing on all the preceding matter have been added. We have not hesitated to use Determinant Notation whenever elegance or simplicity could be gained thereby. This notation has now become so generally known as to

render apology for its use early in the work quite unnecessary.

The Numerical Solution of Triangles, treated of in Chapters IV. and V., has received much attention, each Case being treated of in detail. In connexion with these Numerical Examples, we must acknowledge our obligations to Mr. ROBERT BAILE, M.A., Athlone, who carefully worked and verified them all. They have also been independently verified and tested, and we therefore trust they will all be found correct to the nearest half second.

Geometrical proofs of many Propositions have been added to those commonly given in text-books, *e. g.* the Analogies of Napier and Delambre.

Most of the examples have been taken from University and Science and Art Examination Papers. Many, however, as far as we are aware, appear now for the first time. A series of examples has been appended in the form of Examination Papers.

Among published works on the subject, we are indebted chiefly to those of Todhunter, Snowball, and Luby.

WM. J M'CLELLAND
THOMAS PRESTON.

May, 1885.

PREFACE

TO

SECOND EDITION OF PART I.

IN the preparation of this volume the Authors have very closely adhered to the text of the First Edition. A few Propositions of fundamental importance, which previously appeared amongst the Examples, are given more prominence by being placed in the text; and some articles have been re-written and amended. The Examples are retained, with few alterations, and the Answers to them have been carefully revised.

June, 1887.

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CHAPTER I.

THE SPHERE.

1. A Sphere is a figure such that all points of its surface are equally distant from a certain point within it, called the *Centre*.

Any straight line drawn from the centre of a sphere to the surface is called a *Radius*, and any straight line drawn through the centre and terminated both ways by the surface is called a *Diameter*.

A sphere may be generated by the revolution of a semicircle round its diameter.

2. Great and Small Circles.—Let a sphere be gene-

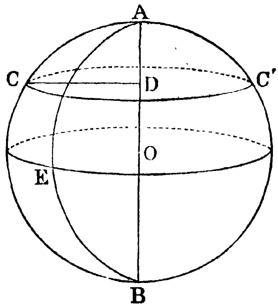

Fig. 1.

rated by the revolution of the semicircle *ACB* (fig. 1)

round its diameter AB. Let C be any point on the semicircle, and let CD be a perpendicular from C on AB.

It is obvious that as ACB revolves round AB, the point C describes a circle round D as centre; and also that O, the middle point of AB, being equally distant from all points on ACB, is the centre of the sphere.

When the plane of the circle CC' described by C passes through O it divides the sphere into two equal parts, and the curve of section is called a *Great Circle.*

When its plane does not pass through the centre it is called a *Small Circle.*

Example.

The Meridians and Equator are great circles. The Parallels of Latitude are small circles.

Spherical Radius.—The angular distance AC is called the *Spherical* or *Angular Radius* of the circle CC'. It is obvious that the spherical radius of a great circle is a quadrant.

3. **Unique Properties of the Great Circle.**—(1). Only *one Great Circle* can be drawn through two given points on the surface of a sphere; for its plane must also pass through the centre; and three points not in the same right line are sufficient to determine a plane completely.

If the two given points be diametrically opposite, the right line joining them passes through the centre of the sphere, and an infinite number of great circles can be drawn through them—as, for example, the meridians on the surface of the Earth.

(2). The *shortest distance* that can be traced on the surface of a sphere between two points on it is the arc of the great circle passing through them.

For of all the circular arcs which can be drawn through two given points A and B [fig. 1 (a)], obviously that arc which has the least curvature is the shortest, since it lies inside the others, and most nearly approaches the right line joining A and B. Thus the shortest arc is that which

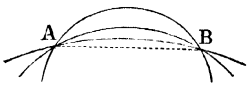

Fig. 1 (a).

belongs to the circle of greatest radius; but the circle of greatest radius which can be drawn on a sphere is the great circle. Therefore, of all the arcs which can be drawn between two points on the surface of a sphere, the great circle arc is the shortest.*

4. **Axes and Poles.**—The line AB (fig. 1) is called the *Axis* of the circle CC' (described by *any* point C of the

* The following method of looking at this question is also instructive:—
If a string be stretched between two points on the surface of a sphere (or on any surface) it will evidently be the shortest distance that can be traced on the surface between the points, since by pulling the ends of the string its length, between the points, will be shortened as much as the surface will permit. Now any part of the string being acted on by two terminal tensions, and by the reaction of the surface, which is everywhere normal to it, must lie in a plane containing the normal to the surface. Hence, the plane of the string contains the normals to the surface at all points of its length; *i.e.* the string lies in the form of a great circle.

semicircle ACB during its revolution round AB), and the extremities A and B of its axis are called its *Poles*. The nearer point A is generally particularized as *the Pole* of the Circle. Any point, and the great circle of which it is the pole, are termed *pole* and *polar*, with respect to one another. It is obvious, from the manner in which CC' was generated, that AB is perpendicular to the plane of CC', for CD remained perpendicular to AB during its entire revolution.

Hence the *Axis* of a circle may be defined as *the diameter of the sphere perpendicular to the plane of the circle*, or *the line joining the centre of the sphere to the pole of the circle.*

Cor.—The pole of a circle is equidistant from all points on the circumference of the circle.

For $$AC^2 = AD^2 + CD^2 = \text{constant}.$$

In the case of a great circle, D becomes the centre of the sphere, and hence the poles of a great circle are equidistant from its circumference.

5. **Primary and Secondary Circles.**—Any circle is called a *Primary* in relation to those *great circles* which cut it at right angles. These latter are called *Secondaries;* e.g. parallels of latitude are primaries, and the meridians are secondaries to them.

In fig. 1, regarding the circle CC' as a primary, the circle ACB during its revolution round AB is in all positions a secondary to CC'. Hence it follows that—

(1). The plane of any secondary contains the axis of the primary.

(2). All the secondaries pass through the poles of the primary.

(3). The planes of all the secondaries have a common line of intersection, viz. the axis of the primary.

(4). If there can be drawn common secondaries to two circles, the planes of those circles are parallel. For, by (3), the two circles have the same axis.

The distance of any point on the surface of a sphere from a circle traced thereon is measured by the arc of the secondary intercepted between the point and the circle.

Example.

The latitude of a place is measured by the arc of the meridian intercepted between the place and the Equator.

6. **The Angle between Two Planes** is measured by the angle between any two lines drawn, one in each plane, perpendicular to their line of intersection.

The angle between any two circles, great or small, is measured by the angle between the tangents drawn to them at their point of intersection.

The Angle between Two Great Circles is equal to—

(1). *The angle between their planes.*

For the tangents to them at their point of intersection lie in their respective planes and are perpendicular to their common diameter.

(2). *The arc intercepted by them on the great circle to which they are secondaries.*

For if O be the centre of a sphere, and AC and BC (fig. 2) secondaries to AB, they are perpendicular to AB, and are each quadrants; therefore OA and OB are per-

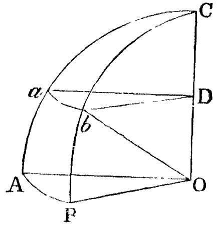

Fig. 2.

pendiculars to OC; hence the angle between OA and OB, or the arc AB, is the angle between the planes of AC and BC.

(3). *The angular distance between their poles.*

For the angular distance between the poles of two circles is the same as the angle between their axes; but the axes are perpendicular to the planes of the circles, and the angle between the perpendiculars to two planes is equal to the angle between the planes.

Hence, by drawing a great circle parallel to a small circle, we see that the angle between the planes of a great and a small circle, or the angle between the planes of two small circles, is equal to the angular distance between the poles of the circles.

7. *To compare the length of the arc of a small circle subtending any angle at its centre with the arc of a great circle subtending an equal angle at the centre of the sphere.*

Let ab (fig. 2) be an arc of a small circle subtending any angle at its centre D, and let AB be an arc of a great circle subtending an equal angle at its centre O, then if the circles be placed parallel, so that they may have a common pole at C, O will be the centre of the sphere, AC and BC will be quadrants, CO will be perpendicular to the planes ABO and abD; and since the perimeters of circles are to each other as their radii, we have

$$\frac{ab}{AB} = \frac{bD}{BO} = \frac{bD}{bO} = \sin bOD = \sin bC = \cos bB.$$

The above equation expresses the length of the arc of a parallel of latitude in terms of the latitude of the place and the corresponding arc of the Equator. Thus $ab = AB$ cos (latitude); or, *the distance* (ab) *between two places in the same latitude, measured on the parallel, is equal to the difference of longitude* (AB) *multiplied by the cosine of the latitude.*

8. **The Lune.**—A *Lune* is a portion of the surface of a sphere enclosed by two great circles.

In fig. 1 the area $ACBEA$ is a lune; so also is the area $AEBC'A$.

The angle between the two great circles bounding the lune is called the *Angle of the Lune.*

The area of a lune is readily expressed in terms of the angle of the lune and radius of the sphere. For we have evidently—

Area of Lune : Area of Sphere :: Angle of Lune : 2π;
or, denoting the angle of the lune by A, and assuming*

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* See Part II., Art. 97.

the area of the sphere to be $4\pi r^2$, where r is the radius of the sphere,

$$\text{Area of Lune} = 2Ar^2.$$

9. **Theorem.**—*Any section of the surface of a sphere by a plane is a circle.*

For (fig. 1) if OD be let fall from O, the centre of the sphere, perpendicular to the plane CC', we have

$$CD^2 = CO^2 - OD^2 = \text{constant},$$

since CO is constant, and OD is fixed. Hence, every point of the section of the surface is equidistant from D, and therefore lies on a circle round D as centre.

10. *The intersection of two spherical surfaces is a circle.*

Consider the triangle ABX formed by joining the centres A and B of the two spheres and any point X on their curve of intersection. The sides of this triangle are *given*, and the line of centres AB is fixed; hence the opposite vertex X (*i.e.* the variable point on the curve of section of the two spheres) is at a constant distance from AB, or lies on a circle having AB for axis.

This property is more general than that stated in the previous Article, and reduces to it as a particular case by supposing the radius of either sphere to become infinitely great.

11. **Solid (or Conical) Angles.**—When three or more planes intersect at a point they enclose what is termed a *Solid Angle* at that point. The much more expressive term, *Conical Angle*, is also used. There is a solid angle, for example, at each corner of a cube, and at each

vertex of a tetrahedron or pyramid. Every closed curve, plane or tortuous, subtends a solid angle at any point in its neighbourhood, the solid angle being enclosed by the lines drawn from the point to every point of the contour of the curve.

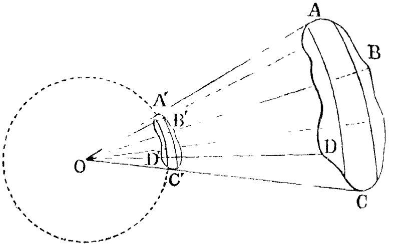

Fig. 3.

Let $ABCD$ (fig. 3) be any curve or plane area. From any point O draw an infinite number of lines OA, OB, OC, OD, &c., passing through the boundary of $ABCD$. These lines form an irregular cone, having its vertex at O, and enclosing the curve or area $ABCD$. The solid angle of this cone is the solid angle which $ABCD$ subtends at O.

To measure this angle, an expedient similar to that employed in reckoning plane angles by circular measure is used. About O describe a sphere of unit radius, and let the cone formed by the lines from O to $ABCD$ intersect the surface of this sphere in the curve $A'B'C'D'$. The curve $A'B'C'D'$ will enclose an area on the sphere, and this area measures the solid angle at O, just as the

arc of a circle of unit radius measures the angle it subtends at the centre. The student must be careful, however, not to regard a solid angle as an *area*, but as a mere *number*, like the circular measure of a plane angle. For as the circular measure of a plane angle is the ratio of the length of the arc subtending it to the radius of the circle, so the ratio of the area $A'B'C'D'$ to the square of the radius of the sphere is the true measure of the solid or conical angle. When the radius of the sphere is unity this ratio becomes numerically equal to the number of units of area enclosed by $A'B'C'D'$.

Since the radius of the sphere is unity, its area is 4π; hence the sum of all the solid angles round any point is 4π, and the solid angle subtended at O by the $1/n^{th}$ part of the surface of the sphere is $4\pi/n$, which is thus a mere number. For example, the solid angle contained by three mutually rectangular planes, as at a corner of a cube, is $\frac{1}{2}\pi$.

Examples.

1. If O is situated inside a closed surface, the sum of the solid angles subtended at O by all the elements of the surface is 4π.

2. If in Ex. 1 O is situated on the surface, the sum of all solid angles is 2π. [For in this case the lines joining O to all the points of the surface intercept half the area of the sphere.]

3. As a case of Ex. 2, the solid angle subtended at any point by an infinite plane, or by any plane curve at a point enclosed by it, is 2π.

4. The solid angle of a right cone of semi-vertical angle α is $2\pi(1 - \cos\alpha)$. [*See* Art. 97.] Thus the solid angle subtended by a circle of radius r at a point on its axis, distant z from the plane of the circle, is $2\pi\left(1 - \frac{z}{\sqrt{r^2 + z^2}}\right)$.

12. **Theorem.**—*The point of intersection of two great circles and the arc joining their poles are pole and polar with respect to one another.*

For let A and B be two points, LO and MO their

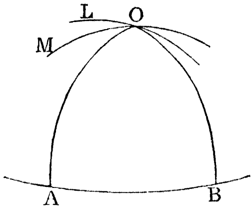

Fig. 4.

polars, O the point of intersection of the polars; then $AO = \frac{1}{2}\pi$, $OB = \frac{1}{2}\pi$; hence OA and OB are secondaries to AB; therefore, &c. [Art. 5 (2).]

Examples.

1. Two great circles bisect each other.

They have a common diameter.

2. The axis of any circle is determined by the intersection of two of its secondaries. [Art. 5 (3).]

3. Find the locus of the centres of a system of circles having common secondaries. *Ans.* The common axis of the system.

4. The pole of any great circle is ninety degrees from the circumference. [Art. 4.]

5. The arc drawn through the poles of two great circles cuts both at right angles.

5*a*. Draw a great circle cutting two given circles, great or small, at right angles.

6. If three or more great circles are concurrent, their poles are concyclic.

7. Any great circle is the locus of the poles of all its secondaries.

8. If from any point on the surface of a sphere *two* great circles can be drawn perpendicular to a given great circle, that point is the pole of the circle.

[For, each of the arcs drawn perpendicular are secondaries; therefore, &c. Art. 5, (2).]

9. The external bisector of an angle passes through the pole of the internal bisector; and conversely.

10. If two great circles are equally inclined to a third, their poles are equidistant from the pole of the third.

10*a*. If a point is equidistant from three great circles, it is also equidistant from their poles.

11. Two equal small circles are drawn touching each other: show that the angle between their planes is twice the complement of their spherical radius. —(*Science and Art Examination Papers.*)

[For, join the centres of the small circles to the centre of the sphere, and also to the point of contact of the circles.

Then the lines drawn to the centre of the sphere contain an angle equal to twice the spherical radius of either small circle, and the lines drawn to the point of contact of the small circles contain an angle equal to the inclination of the planes of the circles; therefore, &c. —(Euc. III. XXII.)]

12. On a sphere whose radius is r a small circle of spherical radius, θ, is described, and a great circle is described having its pole on the small circle; show that the length of their common chord is

$$\frac{2r}{\sin\theta}\sqrt{-\cos 2\theta}.$$

—(*Science and Art Examination Papers.*)

[For the angle between the axis of the small circle and the plane of the great circle is evidently $90 - \theta$. Hence the distance, d, from the middle point of the common chord to the centre of the small circle is given by the equation

$$d = r\cos\theta\cot\theta.$$

Again, if c be the common chord, it follows that

$$\left(\frac{c}{2}\right)^2 = (r\sin\theta + d)(r\sin\theta - d),$$

since $r \sin \theta$ is the radius of the small circle; therefore

$$c = \frac{2r}{\sin \theta} \sqrt{-\cos 2\theta},$$

the negative sign occurring under the radical, since 2θ must be greater than $90°$ for *real* section.]

NOTE—Hence the segments of a great circle, made by the intersection of a small circle of given radius, may be calculated on a sphere of given radius.

13. The angle subtended at the centre of a circle, great or small, by two points on it, is equal to the angle subtended by them at its pole. [Art. 6.]

14. If through the centre O of a sphere a line be drawn parallel to a chord AB of a semicircle ABC described thereon, it meets the sphere at the middle point of the arc BC.

15. Given A, B, C, D, four points on a segment of a circle; prove the relations—

$$\alpha°.\ \sin BC \sin AD + \sin CA \sin BD + \sin AB \sin CD = 0.$$

$$\beta°.\ \sin BC \cos AD + \sin CA \cos BD + \sin AB \cos CD = 0.$$

$$\gamma°.\ \sin \frac{BC}{n} \sin \frac{AD}{n} + \sin \frac{CA}{n} \sin \frac{BD}{n} + \sin \frac{AB}{n} \sin \frac{CD}{n} = 0.$$

CHAPTER II.

SPHERICAL TRIANGLES.

13. In the present chapter it will be shown that geometrical figures drawn on the surface of a sphere (or different spheres of equal radii) have many properties and relationships analogous to those of corresponding figures described on a plane surface; and with the latter the reader is supposed to be already familiar.

14. **Definitions.**—By a *Spherical Figure* is meant a portion of the surface of a sphere enclosed by arcs of *great** circles, *e.g.*

A *Spherical Triangle* is bounded by the arcs of three great circles; a *Spherical Quadrilateral* by four great circles; a *Spherical Polygon* by many great circles.

In the particular case, when the number of enclosing arcs becomes indefinitely great, the figure becomes a *Spherical Curve.* A great circle drawn through two indefinitely near points on a curve is a *Tangent Circle* to the curve. Great and small circles are examples of spherical curves.

The arcs are generally spoken of as the *Sides*, and their angles of inclination as the *Angles* of the spherical figure.

Since, however, three great circles on a sphere intersect one another so as to form *eight* triangles, for the sake of convenience, and to avoid ambiguity, that particular triangle is selected which has two, or if possible three, sides, each less than a quadrant.

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* Unless otherwise stated.

The three sides and three angles are termed the *r*[illegible] of a spherical triangle; and, as will be hereafter seen, any three whatever given in magnitude are necessary and sufficient to determine the remaining three.

15. **By the Arc connecting two Points** on a sphere, unless otherwise expressed, we mean the lesser segment of the great circle passing through the points. (See Chap. I. Art. 3.)

Examples.

1. Each side of a spherical triangle is less than two right angles.

2. Each angle of a spherical triangle is less than two right angles.

[For if possible let ABC be a spherical triangle, having an angle at A greater than two right angles. Continue the arc BA to the point X on the side BC.

Then BX is a semicircle; hence $BX + CX$ is greater; therefore ABC is not a spherical triangle.]

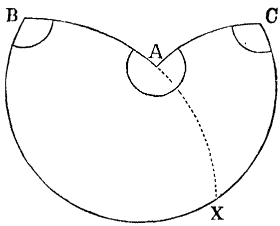

Fig. 5.

3. The area of any spherical triangle is less than $2\pi r^2$. [See Art. 8.]

16. **Colunar or Associated Triangles.**—By producing the sides of a triangle to meet below the base, a triangle is formed, having two sides and the opposite angles respectively supplemental to two sides and the opposite angles of the original triangle, while the remain-

angle of the one is equal to the remaining angle of the other (being opposite angles of a lune). Triangles whose parts are so related are termed *Colunar*, and the three triangles colunar with the given one are sometimes termed its *Associated Triangles.* The triangle taken in the first instance is called the *Primitive Triangle.*

Examples.

1. The three associated triangles, taken in pairs, have a side and an opposite angle of the one equal to a side and an opposite angle of the other.
2. When a triangle is equilateral, the colunars are each isosceles.
3. The arcs joining the vertices of the colunar triangles are equal to the sides of the primitive triangle.

17. **Parts of a Spherical Triangle.**—Let ABC be a spherical triangle described on a sphere having its centre at O. Then $OA = OB = OC =$ the radius of the sphere. The arcs BC, CA, and AB are proportional to the angles BOC, COA, and AOB, which they subtend at the centre,

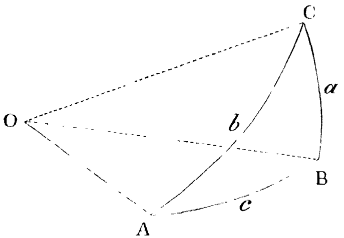

Fig. 6.

and the angles BAC, ABC, and BCA between the sides are equal to the angles between the planes containing the solid angle at O. (Chap. I., Art. 6.)

The notation for the parts of a spherical triangle adopted throughout will be that of Fig. 6, viz. for the three angles the letters A, B, and C, and for the opposite sides the small letters a, b, and c, respectively; though, as will be hereafter noticed, the letters A, B, and C, may denote indifferently the number of degrees in the angles or the circular measure of the angles.

Any expression involving one or more parts of a triangle is a *Function* of these parts, and may be represented in a general form such as—for the side a, $f(a)$; for the sides a, b, and c, $f(a, b, c)$; for two sides b and c, and the angle A, $f(b, c, A)$; for all the parts, $f(a, b, c, A, B, C)$.

18. **Sphere of Infinite Radius.**—At the point of intersection of two great circles, let two tangents be drawn to them in their respective planes; the plane containing both tangents is, with respect to the sphere, the *Tangent Plane* at the point. Now consider the radius of the sphere to increase indefinitely. It is manifest that as the radius increases, each element of surface, and, as a consequence, the whole surface in the neighbourhood of the point, approaches nearer and nearer to the plane; and the great circles described thereon approach nearer and nearer to the tangents. In the limiting case, when the radius of the sphere becomes indefinitely great, and the curvature therefore indefinitely small, the great circles approximate so closely in position to the tangents that they may be regarded as coincident with one another.

Thus the sphere and the great circles degenerate respec-

tively into two* planes (one of which is at infinity), and two pairs of lines (one pair being also at infinity).† But it is to be observed that the angle of inclination of the planes of the great circles alone remains unaltered (*cf.* Chap. I., Art. 6).

Hence the relations involving certain parts of a spherical triangle may be regarded as more general than those involving similar parts of a plane triangle; and by the aid of the expansions

$$\sin a = \frac{a}{r} - \frac{1}{\underline{|3}}\frac{a^3}{r^3} + \frac{1}{\underline{|5}}\frac{a^5}{r^5} - \ldots \text{ to infinity,} \qquad (1)$$

and

$$\cos a = 1 - \frac{1}{\underline{|2}}\frac{a^2}{r^2} + \frac{1}{\underline{|4}}\frac{a^4}{r^4} - \ldots \text{ to infinity} \qquad (2)$$

(see *Plane Trigonometry*), any function of A, B, C, a, b, c, the parts of the former, can be transformed into a function of A, B, C, α, β, γ, the parts of the latter. It should, however, be clearly understood, that α represents the *length* of an arc subtending an angle a, at the centre of

* In accordance with analytical geometry; but it is quite sufficient to understand that any finite portion of the surface is a plane.

† Reference to the Earth, considered as a large sphere, will elucidate this statement.

The surface of a lake or any small portion of water is apparently a perfect plane, although it partakes of the general curvature of the Earth, which may be observed in the case of the ocean, where a distinct curvature is exhibited. It can therefore be readily imagined that if the radius of the Earth were greatly increased, even the surface of the ocean would present no perceptible curvature.

Now take any point on the surface, and draw a plane passing through it

a sphere of radius r; and when r becomes infinite the arc a becomes a right line, viz. a side of the corresponding plane triangle.

It thus appears that every great circle forming part of a diagram in Spherical Geometry will be represented by a right line in the corresponding diagram *in plano;* but we must carefully remember that every right line of a plane figure has not necessarily a representative great circle in the corresponding figure on the sphere; for it will appear hereafter that in some cases small circles on the sphere also become right lines *in plano.*

For example, it is well known that when the base and area of a plane triangle are given, the locus of its vertex is a right line parallel to the base; but when the base and area of a spherical triangle are given, the locus of its vertex is a *small circle* passing through the points diametrically opposite to the extremities of the given base. (See Art. 101.) A moment's consideration will make it plain that the small circle locus here mentioned should *in plano* be represented by a right line, and still further

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and through the centre of the Earth. This plane will cut the surface of the Earth in a great circle (Chap. I., Art. 9). But the surface, at any point, being sensibly a plane, the trace of this great circle on it will be approximately a right line, since the intersection of two planes is a right line. If another great circle be drawn through the point, the two will not differ sensibly from a pair of right lines intersecting at it; and since they both pass through the diametrically opposite point, they will intersect at it also as a pair of right lines, the surface of the Earth being at this point also approximately a plane. Now if the radius of the Earth were indefinitely increased, no curvature whatever would be observed at any part of its surface, and the point diametrically opposite to any selected one would be infinitely distant from it, and the results stated above would follow.

a right line parallel to the base of the triangle. For it passes through the points diametrically opposite to the extremities of the base, and *in plano* these points are infinitely distant; hence the small circle in question becomes one of infinite radius, that is, a right line; and it must meet the given base at infinitely distant points, that is, it must be parallel to it. Hence we are not to conclude that all rectilinear loci in Plane Geometry are represented by great circle loci on the sphere; but, on the contrary, that any small circle on the sphere, constrained to pass through a point diametrically opposite to any point of a spherical figure, is represented by a right line in the corresponding plane figure.

Another example illustrating this principle will be found in Art. 169, *Cor.* 6.

19. **The Analogy between Problems in Plane and Spherical Trigonometry** referred to in the previous Article may be more clearly illustrated as follows:—

(1). *Any two sides of a triangle are together greater than the third side.*

[Proof as *in plano*, *v.* Euc. I. xx. Otherwise thus:— The great circle AB (fig. 6) is the shortest distance measured on the sphere between the points A and B; therefore

$$AC + BC > AB. \quad \text{(See also Euc., XI. xx.)]}$$

(2). *Only one triangle can be constructed with three given arcs (any two of which are greater than the third).*

[If we attempt to construct a triangle with three given arcs, as in Euc., I. XXII., we obtain two triangles on the same

base with their vertices on opposite sides of it. These triangles cannot be shown equal by direct superposition* on account of the curvature of their surfaces, but they may be divided into others which can be superposed (see Note, Art. 100); and it is also evident that one of them can be placed on the sphere with its vertices diametrically

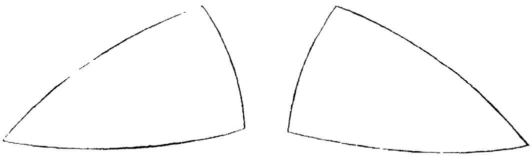

Fig. 7.

opposite to the vertices of the other. In this position it is clear that the angles of one are equal to the angles of the other; for their sides are parts of the same three great circles, and their angles are the angles between the planes of these circles, or the angles of equal lunes. Two such triangles are represented in fig. 7, and are said to be *symmetrically equal.*

(3). *If two triangles have two sides, and the included angle of one respectively equal to the two sides and the included angle of the other, the triangles are equal in every respect.*

[Euc., I. iv.]

[For two sides of given length intersecting at a given angle determine the vertices of the triangle. Therefore (Art. 3 (1)) the base and remaining angles are determined, and the triangle is unique.]

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* If the triangles be cut out from a thin flexible spherical shell, they can be superposed directly by reversing the curvature of one of them.

(4). *The angles at the base of an isosceles triangle are equal to one another.**

[Proceed exactly as in Euc., I. v., using (3)].

(5). *If two angles of a triangle be equal, the triangle is isosceles.*

(6). *The sum of one pair of opposite angles of a quadrilateral inscribed in a circle is equal to the sum of the remaining pair.*†

[For by joining the pole of the circle to the angles of the quadrilateral we have four isosceles triangles; therefore, &c., by (4).]

(7). *The sum of the sides of a quadrilateral is greater than the sum of the diagonals.*

[Apply (1).]

(8). *The greater side of every triangle is subtended by the greater angle.*

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* Otherwise thus :—Let ABC be an isosceles triangle on a sphere, with centre O. At A and B draw tangents to the equal sides CA and CB, in their respective planes. These tangents will intersect OC, produced, in the same point T. For if possible let the tangent at B intersect it in T'. Then in the two triangles AOT and BOT' we have the angles TOA and $T'OB$ equal, the angles OAT and OBT' right angles, and $OA = OB$; therefore (Euc., I. xxvi.) $OT = OT'$.

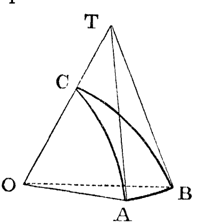

Fig. 6 (*a*).

Also $AT = BT$, and the tangents to AB at its extremities are also equal; therefore, &c., Euc. I. viii. This being established independently, (2) and (3) follow from it.

† This property may be regarded as the criterion of a cyclic spherical quadrilateral. (See Art. 25*a*.)

For, let ABC (fig. 8) be the given triangle, having the angle B greater than the angle A. Draw a great circle

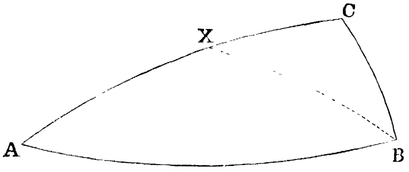

Fig. 8.

BX, making the angle ABX = angle A. Then $BX = AX$. But $BX + CX > BC$; therefore $AC > BC$.

It may be easily shown that the theorems given in Euc. I., Props. XV., XVIII., XXI., XXIV., XXV., XXVI., are more generally true for the sphere.

(9). *The internal bisectors of the angles of a triangle, or two external and one internal, are concurrent.* [(Euc. IV. v.)] [The points of concurrence are the poles of the small circles touching the sides of the triangle.]

(10). *The arcs of great circles bisecting the sides of a triangle at right angles are concurrent.* [(Euc. IV. v.)]

[The point of concurrence is the pole of a small circle passing through the vertices of the triangle.]

(11). *Given the base and sum of base angles, the external bisector of the vertical angle always passes through a fixed point.* (Dublin University Exam. Papers.)

For, let ABC (fig. 9) be a triangle having the given base AB. Upon AB construct an isosceles triangle AOB, having each of the base angles OAB and $OBA = \frac{1}{2}(A+B)$.

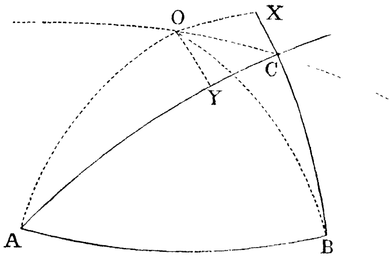

Fig. 9.

Draw OX and OY secondaries to BC and AC, respectively.

In the two right-angled triangles BOX and AOY, $AO = BO$, $\angle\ OAY = \angle\ OBX = \frac{1}{2}(B-A)$; therefore $AY = BX$, and $OX = OY$.

Again, compare triangles COX and COY: $OX = OY$, OC is common, and the angles at X and Y are right angles; hence CO is the bisector of the external vertical angle passing through the fixed point O above determined.

(12). Given of a triangle the base and *sum* or difference of base angles, the *internal* and *external bisectors* of the vertical angle pass through fixed points.

[Cf. Prop. (11).]

20. **Theorem.**—*The great circle bisecting the sides of a triangle intersects the base* 90° *distant from its middle point.*

Let ABC (fig. 10) be the triangle; X, Y, Z, the middle points of the sides; AL, BM, and CN, secondaries drawn through the vertices of the triangle to the great circle XY, passing through the middle points of the sides. Let XY

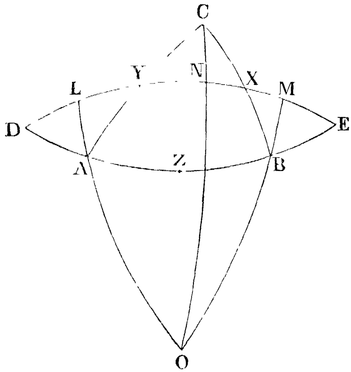

Fig. 10.

meet the base in the points D and E, which are, therefore, diametrically opposite. O is the pole of XY. (Chap. I. Ex. 8.)

Now in the two triangles ALY and CYN we have $AY = CY$, $\angle AYL = \angle CYN$, and $\angle ALY = \angle CNY$; therefore the triangles are equal in every respect. Hence $AL = CN$. Similarly it can be shown that $BM = CN$.

Therefore

$$AL = BM = CN.$$

Again, in the two triangles ALD and BME it follows easily that $DL = ME$ and $AD = BE$; hence

$$DZ = ZE = 90°.$$

*Cor.—The perpendiculars of a Spherical Triangle meet in a point.**

[Regarding XYZ (fig. 10) as the triangle, the perpendicular from Z on XY is, by the above, the polar of the point E. It is, therefore, perpendicular to AB. Similarly the perpendiculars from X and Y on the sides YZ and ZX are at right angles to BC and CA, respectively; therefore, &c., Art. 19 (10)].

Exercises on Fig. 10.

1. The arcs DL and XY are complementary.
2. The triangle AOB is isosceles.
3. The angle $LAY = S - A$, where $2S = A + B + C$.
4. The arcs OC and OA are supplementary.
5. Given the base and the sum of the three angles, the locus of the vertex C is a small circle having its pole at O.

[For under the given conditions the triangle AOB is fixed; hence also OC, by the aid of Ex. 4; therefore, &c.]

6. D and E are the middle points of the sides of the colunars on AC and BC respectively.

Polar Triangles.

Definition.—Two triangles so related that the vertices of the one are the poles of the sides of the other are called, with respect to one another, *Polar Triangles.*

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* The proof here given is due to Sadlier. See also Art. 54.

21. **Reciprocal Relations of the Sides and Angles of Polar Triangles.**—*If two triangles be so related that the vertices of the one are the poles of the corresponding sides of the other; then, conversely, the vertices of the latter shall be the poles of the corresponding sides of the former.*

Let the points A, B, and C (fig. 11), be respectively the poles* of $B'C'$, $C'A'$, and $A'B'$ the sides of the triangle $A'B'C'$. Produce the side $B'C'$ (if necessary) to meet AC in the point M, and AB in the point N.

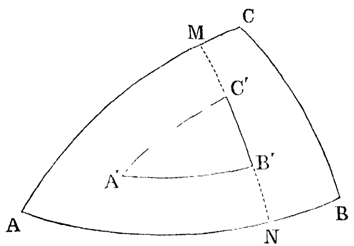

Fig. 11.

Now, since A is the pole of $B'C'$, the arc AB' is a quadrant; and since C is the pole of $A'B'$, the arc CB' is a quadrant; hence B' is the pole of AC: similarly, A' is the pole of BC, and C' the pole of AB; therefore, &c.

[Art. 12.]

Cor. 1.—The arcs connecting the corresponding vertices of a triangle and its polar pass through a point, viz. the common *Orthocentre* of the two triangles.

* The poles of the sides of a triangle may be conveniently obtained by drawing great circles through the vertices perpendicular to the opposite sides, and by taking points on them (measured from the sides towards the vertices) 90° distant from the sides. These points are evidently the required poles.

[For the arc joining A and A' (fig. 11) is perpendicular to each of the arcs BC and $B'C'$, since it passes through their poles; therefore, &c., Art. 20, *Cor.*].

Cor. 2.—The sides of a triangle intersect the corresponding sides of the polar triangle in six points lying on a great circle.

[For the arc AA' is the polar of the points of intersection of BC and $B'C'$: similarly the arcs BB' and CC' are the polars of the points of intersection of CA, $C'A'$ and AB, $A'B'$; therefore, &c., by *Cor.* 1.]

22. **Theorem.**—*The angles of the primitive triangle are the supplements of the corresponding sides of the polar; and conversely.*

Since B' (fig. 11) is the pole of AC, $B'M$ is a quadrant, and for a similar reason $C'N$ is also a quadrant (Chap. I., Ex. 4).

Hence $MN + B'C' =$ two right angles.

But MN is equal to the angle A; therefore the angle A and the side $B'C'$ are supplemental.

Similarly, the angles B and C are the supplements of the sides $C'A'$ and $A'B'$ respectively; therefore, &c.

Note.—From the above property, Polar Triangles are also termed *Supplemental Triangles;* and if a', b', c', A', B', C', represent the parts of the triangle $A'B'C'$ (fig. 11), it follows that

$$A + a' = B + b' = C + c' = \pi.$$

Also

$$A' + a = B' + b = C' + c = \pi.$$

These results are of the greatest importance, inasmuch as any theorem which holds good between the sides and angles (e.g. a, b, c, A, B) of a spherical triangle necessarily involves a reciprocal or supplemental theorem, involving the opposite angles and sides (viz. A, B, C, a, b), and which may be derived from it by changing the sides into the supplements of the corresponding angles, and the angles into the supplements of the corresponding sides.

23. The results obtained in the previous Article may be further exemplified, as follows:—

(a). It has been seen that two sides of a triangle are greater than the third: thus $b + c > a$.

Applying this inequality to the sides of the supplemental triangle, it follows that

$$\pi - B + \pi - C > \pi - A.$$

Hence, $$B + C - A < \pi, \text{ i.e. } S - A < \frac{\pi}{2}. \qquad (1)$$

Similarly, $$C + A - B < \pi, \text{ i.e. } S - B < \frac{\pi}{2}. \qquad (2)$$

And $$A + B - C < \pi, \text{ i.e. } S - C < \frac{\pi}{2}. \qquad (3)$$

Or, *the difference between any two angles of a spherical triangle is less than the supplement of the third angle.*

Particular cases of the inequalities (1), (2), and (3), are worthy of notice.

a. Suppose, for example, that the primitive triangle is right-angled at C, it follows from (3) that

$$A + B < \frac{3\pi}{2};$$

or, *the sum of the angles of a right-angled triangle is less than four right angles.*

β. It likewise follows, by the aid of (2), that

$$A - B < \frac{\pi}{2};$$

or, *the difference of the oblique angles of a right-angled triangle is less than a right angle.*

(*b*). Prop. (4), Art. 19, involves the converse theorem, viz., If two angles of a triangle are equal, the sides subtending them are also equal. [See Prop. (3), Art. 19.]

(*c*). Prop. (8), Art. 19, involves the converse theorem, viz., The greater angle is subtended by the greater side.

(*d*). Prop. (3), Art. 19, involves the supplemental theorem, viz., If two triangles have two angles and a side of the one equal to two angles and a side of the other, the sides being similarly situated with respect to the equal angles, the triangles are equal in every respect.

(*e*). In any triangle, if $C = A + B$, then $C - A$ and $C - B$ are each less than a right angle.

For $C - A = B$, and $C - B = A$; but $B + C - A < \pi$; hence $B < \frac{\pi}{2}$, and $A < \frac{\pi}{2}$; therefore, &c.

Examples.

1. Given two angles of a triangle to be 45° and 120°, find the maximum value of the third angle.

Let A be the third angle, then by aid of (1),

$$A + 120 - 45 < 180°.$$

Hence,

$$A < 105°.$$

2. The angles of a triangle are A, 30° and 150°; find the maximum value of A. *Ans.* $A < 60°$.

[Proof as before.]

3. If the angles are A, 20° and 110°, find the maximum value of A. *Ans.* $A < 90°$.

4. If the difference between any two angles of a triangle is equal to 90°, the remaining angle is less than 90°. [Cf. Art. 23, β.]

5. If the primitive triangle be equilateral or isosceles, the supplemental triangle is equilateral or isosceles.

24. **Theorem.**—*If two sides of a triangle are supplemental, the opposite angles are supplemental.*

For in the triangle ABC (fig. 12), if $AC + AB = \pi$,

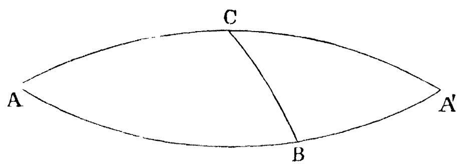

Fig. 12.

since the arc ABA' is a semicircle, then will

$$AC = A'B, \text{ and } A'C = AB.$$

Therefore the two triangles ABC and $A'BC$ are equal in every respect; therefore, &c.

Examples.

1. If two sides of a triangle are supplemental, two sides of the polar triangle are likewise supplemental.

2. In fig. 12, the arc joining the middle point M of BC with A or A' is a quadrant.

[The triangles $A'BM$ and ACM are equal in every respect.]

3. Given the base of a spherical triangle, and the sum of the sides equal to two right angles, find the locus of the vertex.

Ans. A great circle, having the middle point of the base as pole (Ex. 2).

25. **Angular Limits of the Sides and Angles of a Spherical Triangle.**—It has been said that each side and each angle of a spherical triangle may have any values between 0° and π. It is, therefore, manifest that the sum of the angles cannot exceed 3π; and it will be here shown that the sum of the sides cannot exceed 2π. For in the triangle $A'BC$ (fig. 12),

$$BC < A'C + A'B.$$

To each add

$$AC + AB.$$

Hence,

$$BC + CA + AB < AC + A'C + AB + A'B < 2\pi.$$

Hence $a + b + c$ *can have any value between* 0° *and* 2π,
and $A + B + C$ *can have any value between* π *and* 3π.

The latter may be seen otherwise, thus:—

Since 0°, $a + b + c$, 2π, are in ascending order of magnitude; hence, for the supplemental triangle,

$$0°, \quad \pi - A + \pi - B + \pi - C, \quad 2\pi,$$

are likewise in ascending order of magnitude; therefore, &c.

25 (a). **Theorem.**—*If the sum of one pair of opposite angles of a spherical quadrilateral be equal to the sum of the other pair, the quadrilateral is cyclic.*

For let $ABCD$ [fig. 12 (a)] be a quadrilateral, such that the sum of the angles A and C is equal to the sum of the angles B and D; and if possible let the circle passing

through B, C, D not pass through A, but let it, as in

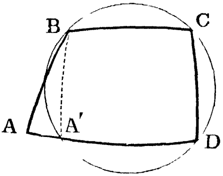

Fig. 12 (a).

Fig. 12 (a), intersect the arc AD in some point A'. Join BA'.

Then, by Art. 19 (6), since $A'BCD$ is in a circle,

$$A' + C = D + A'BC.$$

But

$$A + C = D + ABC.$$

Therefore, by subtracting,

$$A - A' = ABA', \quad \text{or } A - ABA' = A',$$

which is impossible, since A' is the external angle of the triangle ABA'; and, by Art. 23 (a), the difference of any two angles of a triangle is less than the supplement of the third. Therefore the points A, B, C, D lie on the same circle, *i.e.* the quadrilateral is cyclic.

25 (b). **Analogue of Ptolemy's Theorem.**—*If a, b, c, d be in order the sides, δ and δ' the diagonals of a spherical quadrilateral inscribed in a circle; then*

$$\sin \tfrac{1}{2}a \sin \tfrac{1}{2}c + \sin \tfrac{1}{2}b \sin \tfrac{1}{2}d = \sin \tfrac{1}{2}\delta \sin \tfrac{1}{2}\delta'.$$

For if a', b', c', d' be the chords of the arcs which form

the sides of the quadrilateral, e' and f' those of the diagonals, it is clear that a', b', &c., are the sides and diagonals of a plane quadrilateral inscribed in a circle, and therefore

$$a'c' + b'd' = e'f';$$

but $a' = 2r \sin \frac{1}{2}a$, $b' = 2r \sin \frac{1}{2}b$, &c., where r is the radius of the sphere. Therefore, &c.

Examples.

1. Any side of a triangle is greater than the difference between the other two sides.

2. When does the primitive triangle coincide with the supplemental?

Ans. When its sides and angles are each $\frac{\pi}{2}$.

3. Assuming the earth to be a sphere, is the area included between two meridians and a parallel of latitude a spherical triangle? Give a reason for your answer. —(*Science and Art Exam. Papers.*)

4. The sides b and c of a triangle are produced both ways to points x and x', y and y', 90° distant from the middle point of the sides. Show that, if secondaries to the sides from x and y intersect in P, and from x' and y' in P', the points P and P' are diametrically opposite.

5. The triangle $P'Q'R'$ is the polar triangle of that formed by joining the middle points of the sides of ABC, where Q' and R' are formed by similarly producing the sides c and a, a and b, through the extremities of the remaining side.

Note on Art. 26.

[The formula of Art. 26 may be obtained directly by equating the projection of OD on OE to the sum of the projections of OC and CD on the same line. Thus

$$OD \cos c = OC \cos b + CD \cos C \sin b,$$

or

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.]$$

CHAPTER III.

FUNDAMENTAL RELATIONS BETWEEN THE PARTS OF A SPHERICAL TRIANGLE.

SECTION I.

26. *Having given the sides of a spherical triangle, to determine the cosines of the angles.*

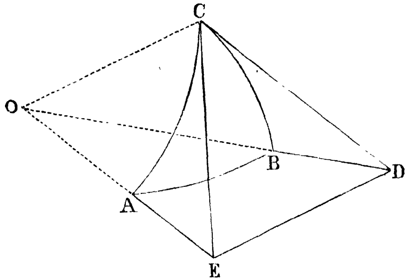

Fig. 13.

Let ABC be a spherical triangle; a, b, and c, its sides; O the centre of the sphere. Draw CD and CE tangents to a and b at C.*

Since these tangents lie in the planes of the circles to which they are drawn, they will meet the radii OB and

* The same result will be obtained if, instead of drawing tangents at C, any point C' be taken on OC, and lines $C'D'$ and $C'E'$ be drawn perpendicular to it, meeting OB and OA in D' and E', respectively.

OA in D and E respectively, and the angle between EC and CD is equal to the angle C (Chap. I., Art. 6).

Now, from the two triangles ECD and EOD we have

$$DE^2 = CD^2 + CE^2 - 2\,CD \,.\, CE \,.\, \cos C \;.\;.\;. \qquad (\alpha)$$

Also $$DE^2 = OD^2 + OE^2 - 2\,OD \,.\, OE \,.\, \cos c \;.\;.\;.; \qquad (\beta)$$

and since the angles OCE and OCD are each $90°$,

$$OC^2 = OD^2 - CD^2 = OE^2 - CE^2.$$

Hence, on subtracting (α) from (β),

$$0 = OC^2 + CD \,.\, CE \cos C - OD \,.\, OE \,.\, \cos c;$$

or,

$$\cos c = \frac{OC}{OD} \cdot \frac{OC}{OE} + \frac{CD}{OD} \cdot \frac{CE}{OE} \,.\, \cos C;$$

$$\left.\begin{array}{ll} \text{therefore,} & \cos c = \cos a \cos b + \sin a \sin b \cos C. \\ \text{Similarly,} & \cos b = \cos c \cos a + \sin c \sin a \cos B, \\ \text{and} & \cos a = \cos b \cos c + \sin b \sin c \cos A. \end{array}\right\} \qquad (1)$$

From these formulæ we obtain the angles in terms of the sides. Thus:

$$\left.\begin{array}{l} \cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c} \;.\;.\;.; \\[2ex] \cos B = \dfrac{\cos b - \cos c \cos a}{\sin c \sin a} \;.\;.\;.; \\[2ex] \cos C = \dfrac{\cos c - \cos a \cos b}{\sin a \sin b} \;.\;.\;. \end{array}\right\} \qquad (2)$$

These results are of special importance, and the student should make himself perfectly familiar with them at the outset.

They apply to all spherical triangles, whether the arcs be greater or less than quadrants.

27. Generalization.—In the preceding Article it is plain, on referring to the figure, that the proof given only applies to a spherical triangle having the sides a and b each less than $\frac{1}{2}\pi$. For, in the two triangles OCE and OCD, each right-angled at C, the angles COE and COD are acute, and therefore the arcs a and b subtended by them on the sphere are less than $\frac{1}{2}\pi$. No such restriction has been placed on the limits of the remaining side c: it may, therefore, be greater, equal to, or less than, $\frac{1}{2}\pi$.

It therefore remains to be shown that the foregoing theorem is applicable when *either of* or *both* the sides a and b are greater than quadrants.

(1). *In the former case* let a be greater than, and b less than, a quadrant.

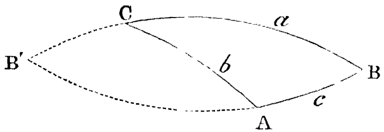

Fig. 14.

Let a and c (fig. 14) be produced to meet in B'. Then in the triangle $AB'C$, by the preceding it follows that

$$\cos AB' = \cos B'C \cos AC + \sin B'C \sin AC \cdot \cos ACB';$$

therefore,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

(2). *In the latter case* let a and b (fig. 15) be each greater than a quadrant

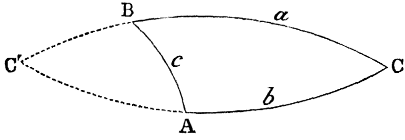

Fig. 15.

Then in the triangle ABC' we have

$$\cos c = \cos BC' \cos AC' + \sin BC' \sin AC' \cos C;$$

or,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

The remaining cases, when either of or both the sides a and b are quadrants, are left as easy exercises for the student. The formulæ referred to are, therefore, applicable to all triangles drawn on the surface of a sphere.

27 (*a*).—**Right-angled Triangle.**—We shall now apply the formula of Art. 26 to a triangle right-angled at C, and deduce the relations existing amongst the parts of such a triangle.

Since $C = \frac{1}{2}\pi$, $\cos C = 0$; hence

(1). $$\cos c = \cos a \cos b.$$

(2). Again, from the formula,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

On substituting the value of $\cos a$ from Ex. (1) we have

$$\cos A = \frac{\tan b}{\tan c}.$$

Similarly,

$$\cos B = \frac{\tan a}{\tan c}.$$

(3). On substituting the value of $\cos b$ from Ex. (1) we have, from (2), after reduction,

$$\sin A = \frac{\sin a}{\sin c}.$$

Similarly,

$$\sin B = \frac{\sin b}{\sin c}.$$

(4). From Examples (2) and (3) we have, by division,

$$\tan a = \tan A \sin b,$$

and

$$\tan b = \tan B \sin a.$$

(5). From (4) we have

$$\tan a \tan b = \tan A \tan B \,.\, \sin a \sin b;$$

or,

$$\cot A \cot B = \cos a \cos b = \cos c.$$

(6). From (2) and (3) we have

$$\cos A = \cos a \sin B.$$

The above six formulæ, connecting the sides and angles of a right-angled triangle, are expressed in general terms thus:—

(1) cos (hypotenuse) = product of cosines of sides.
(2) cos (angle) = tangent (adjacent side) ÷ tan (hypotenuse).
(3) sin (angle) = sin (opposite side) ÷ sin (hypotenuse).
(4) tan (side) = tan (opposite angle) × sin (remaining side).
(5) cos (hypotenuse) = product of cotangents of base angles.
(6) cos (angle) = cos (opposite side) × sin (remaining angle).

(7). Let $b = c = \frac{\pi}{2}$, and the expression

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

reduces to

$$\cos a = \cos A, \qquad \text{[Vide Art. 6, fig. 2.]}$$

as is otherwise evident, since in this case C is the pole of AB.

(8). Let $$a = b = c;$$

then $$\cos a = \cos^2 a + \sin^2 a \cos A;$$

or, $$\cos a (1 - \cos a) = (1 - \cos^2 a) \cos A.$$

Hence, $$\sec A = 1 + \sec a,$$

a constant relation between the angle and side of an equilateral spherical triangle, which shows that when the sides are quadrants the angles are right angles, and conversely.

(9). Given a side or angle of an equilateral triangle; solve it completely, and deduce the limiting values of the sides and angles of an equilateral triangle. [Apply Ex. (8).]

(10). Given the latitudes and difference of longitudes of two places on the Earth's surface, compute the distance between them.

—(*Science and Art Exam. Papers*).

(11). Apply Art. 26 (1) to prove the theorem of Art. 19 (3).

28. Analogous Formula in Plano.—Supposing the radius of the sphere to be *indefinitely great*, show that the formula

$$\frac{\cos a - \cos b \cos c}{\sin b \sin c} = \cos A$$

degenerates into

$$\frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} = \cos A,$$

where A is an angle, and α, β, and γ the sides of a plane triangle.

Expanding $\cos a$, &c., by the aid of formulæ, Chap. II., Art. 18, and neglecting all powers of the radius above the second, we have

$$\frac{\left(1 - \frac{\alpha^2}{\underline{|2}r^2}\right) - \left(1 - \frac{\beta^2}{\underline{|2}r^2}\right)\left(1 - \frac{\gamma^2}{\underline{|2}r^2}\right)}{\frac{\beta}{r} \cdot \frac{\gamma}{r}} = \cos A,$$

which reduces to

$$\frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} = \cos A.$$

29. Legendre's Theorem.—*If the sides of a spherical triangle be small compared with the radius of the sphere, then each angle of the spherical triangle exceeds by one-third of the spherical excess the corresponding angle of the plane triangle, the sides of which are of the same length as the arcs of the spherical triangle.*

In the foregoing example the powers of the radius higher than the *second* were neglected. Suppose, however, a closer approximation to be made, and the powers of $\frac{1}{r}$ above the *fourth* neglected.

Then

$$\cos A = \frac{1 - \frac{\alpha^2}{2r^2} + \frac{\alpha^4}{24r^4} - \left(1 - \frac{\beta^2}{2r^2} + \frac{\beta^4}{24r^4}\right)\left(1 - \frac{\gamma^2}{2r^2} + \frac{\gamma^4}{24r^4}\right)}{\frac{\beta}{r}\left(1 - \frac{\beta^2}{6r^2}\right)\frac{\gamma}{r}\left(1 - \frac{\gamma^2}{6r^2}\right)}$$

$$= \frac{\frac{\beta^2 + \gamma^2 - \alpha^2}{2r^2} + \frac{\alpha^4 - \beta^4 - \gamma^4 - 6\beta^2\gamma^2}{24r^4}}{\frac{\beta\gamma}{r^2}\left(1 - \frac{\beta^2 + \gamma^2}{6r^2}\right)}.$$

But by the aid of the Binomial Theorem we have

$$\frac{1}{1-\dfrac{\beta^2+\gamma^2}{6r}} = \left(1-\frac{\beta^2+\gamma^2}{6r^2}\right)^{-1} = 1+\frac{\beta^2+\gamma^2}{6r^2}, \text{ nearly.}$$

Hence

$$\cos A = \frac{\left\{\dfrac{\beta^2+\gamma^2-\alpha^2}{2r^2}+\dfrac{\alpha^4-\beta^4-\gamma^4-6\beta^2\gamma^2}{24r^4}\right\}\left\{1+\dfrac{\beta^2+\gamma^2}{6r^2}\right\}}{\dfrac{\beta\gamma}{r^2}}$$

$$= \frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma} - \frac{2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4}{24\beta\gamma r^2}.$$

Now consider a plane triangle of sides, α, β, and γ; angles A', B', and C'; and area $= \Delta = \frac{1}{2}\beta\gamma \sin A'$.

The above equation reduces to the form (Plane Trig.)

$$\cos A = \cos A' - \frac{\sin^2 A' \,.\, \beta\gamma}{6r^2}.$$

Let the excess of A over A' be denoted by θ, θ being a very small quantity; then $A = A' + \theta$, and

$$\cos A = \cos A' - \theta \sin A' \text{ nearly};$$

therefore,

$$\cos A' - \frac{\sin^2 A'}{6r^2}\beta\gamma = \cos A' - \theta \sin A';$$

or,

$$\theta = \frac{\Delta}{3r^2}.$$

We have, therefore,

$$A = A' + \frac{\Delta}{3r^2}; \qquad B = B' + \frac{\Delta}{3r^2}; \qquad C = C' + \frac{\Delta}{3r^2}.$$

Whence

$$A + B + C - A' - B' - C' = \frac{\Delta}{r^2},$$

or,

$$A + B + C - \pi = \frac{\Delta}{r^2}.$$

But the expression $A + B + C - \pi$ denotes the excess of the sum of the angles of a spherical triangle over the sum of the angles of a plane triangle (both being expressed in circular measure), and is therefore called the ***Spherical Excess;*** therefore, &c.

[*See also Exam. Paper* VIII. 2, 3; and Art. 107.]

SECTION II.

30. *Expression for the sine of an angle of a spherical triangle in terms of the trigonometrical functions of the sides*

We have (Art. 26),

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

therefore

$$\begin{aligned}
\sin^2 A &= 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2 \\
&= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} \\
&= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} \\
&= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c} \\
&= \frac{4n^2}{\sin^2 b \sin^2 c},
\end{aligned}$$

where

$$4n^2 = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c.^*$$

$$\left.\begin{aligned}
&\text{Hence,} & \sin A &= \frac{2n}{\sin b \sin c}. \\
&\text{Similarly,} & \sin B &= \frac{2n}{\sin c \sin a}, \\
&\text{and} & \sin C &= \frac{2n}{\sin a \sin b}.
\end{aligned}\right\} \qquad (3)$$

---

* The function n has been called the sine of the solid angle that the triangle subtends at the centre of the sphere (*vide* Salmon's *Geometry of Three Dimensions*, Art. 54), and may be written in the determinant form

$$4n^2 = \begin{vmatrix} 1, & \cos c, & \cos b \\ \cos c, & \cdot, & \cos a \\ \cos b, & a, & 1, \end{vmatrix}.$$

31. **Rule of Sines.**—From the value of sin A in the previous Artice we have at once

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{2n}{\sin a \sin b \sin c}. \qquad (4)$$

Hence, the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.*

The corresponding theorem *in plano* is, "The Sines of the Angles of a Plane Triangle are proportional to the opposite Sides."

Examples.

1. Given the base c and the function n, find the locus of the vertex.

—(*Q. U. I., Exam. Papers.*)

Since $$\sin A = \frac{2n}{\sin b \sin c},$$

we have given $\sin b \sin A$, which, by the aid of Art. 31, or 27, Ex. 3, is equal to $\sin p$, where p is the perpendicular from the vertex on the base. The locus is, therefore, a small circle having the same poles as the base.

2. If α, β, γ, be the perpendiculars of a triangle, prove that

$$\sin a \sin \alpha = \sin b \sin \beta = \sin c \sin \gamma = 2n.$$

[For (Art. 27, Ex. 3),

$$\sin \alpha = \sin b \sin C, \quad \sin \beta = \sin c \sin A, \quad \sin \gamma = \sin a \sin B;$$

therefore, &c., Art. 30 (3).]

Note.—The rule of sines follows at once from the equation

$$\sin a = \sin b \sin C = \sin c \sin B;$$

therefore

$$\frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}.$$

Therefore, &c.

---

* Or thus:—take P, the pole of the side c, and apply equation (1), Art. 26, to the triangles ACP and BCP. Then $\cos CP = \sin a \sin B = \sin b \sin A$. A particular case of this theorem has already been proved in Art. 24.

3. In Ex. 2 show that

$$\sin a \cos \alpha = \sqrt{\cos^2 b + \cos^2 c - 2 \cos a \cos b \cos c},$$

with similar expressions for $\cos \beta$ and $\cos \gamma$.

[Since

$$\sin \alpha = \sin b \sin C = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin a}.$$

Hence $\qquad \sin^2 a \sin^2 \alpha$, or $\sin^2 a - \sin^2 a \cos^2 \alpha$

$$= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c,$$

therefore, &c.]

4. The sines of the segments of the base of a triangle made by the bisectors of the internal and external vertical angle are to one another as the sines of the adjacent sides. (Cf. Euc., VI. III.)

5. The bisector of the base divides the vertical angle into two parts, the ratio of whose sines is the inverse ratio of the sines of the adjacent sides.

6. If M be the middle point of the base AB of a triangle ABC, and M' * a point on the base, such that the angle ACM is equal to the angle BCM', prove that

$$\sin AM' : \sin BM' :: \sin^2 b : \sin^2 a.$$

7. If p_1, p_2 be the perpendiculars drawn from the mid-point M of the base of a spherical triangle on the great circle bisectors of the vertical angle A, and p_3 the perpendicular from A on the great circle perpendicular to the base through M, prove—

(1) $$\sin p_1 \sin p_2 = \tfrac{1}{2} \sin p_3 \sin \frac{a}{2} \sin (B + C).$$

(2) Give the analogous theorem for a plane triangle.

—(*Educational Times*, August, 1884.)

[Let x and y be the points where the external and internal bisectors meet the perpendicular through M; then

$$\frac{\sin p_1 \sin p_2}{\sin p_3} = \frac{\sin Mx \sin x \,.\, \sin My \sin y}{\sin Ax \,.\, \sin x} = \frac{\sin Mx \,.\, \sin My}{\sin xy};$$

therefore, &c.

The analogous theorem *in plano* is $\dfrac{4p_1p_2}{p_3} = a \sin A$.]

* The arc CM' is called a *Symmedian* of the triangle ABC.

32. Geometrical Proof of the Rule of Sines. – *To show geometrically that the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.*

Let ABC (fig. 16) be a spherical triangle, O the centre of the sphere. From any point P in OC let fall a perpen-

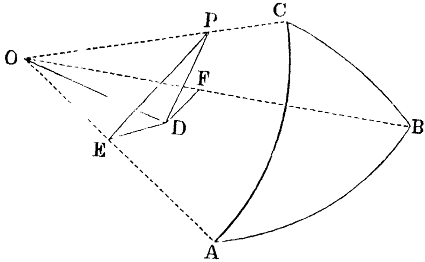

Fig. 16.

dicular PD on the plane AOB. Join OD, and let fall DE perpendicular on OA. Join PE. PD is perpendicular to all lines drawn through D in the plane AOB.

Now

$$OP^2 = OD^2 + PD^2 = OE^2 + DE^2 + PD^2 = OE^2 + PE^2;$$

therefore PEO is a right angle.

Now, since PE and DE are drawn in the planes OAC and OAB perpendicular to their line of intersection OA,

$$\angle PED = \angle A;$$

therefore

$$\sin A = \frac{PD}{PE}, \quad \sin b = \frac{PE}{OP},$$

and

$$\sin A \sin b = \frac{PD}{OP}.$$

Similarly, by letting fall a perpendicular DF on OB we get

$$\sin B \sin a = \frac{PD}{OP} = \sin A \sin b;$$

hence

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

33. **Expressions for the sine, cosine, and tangent of half an angle of a triangle as functions of the sides.**

We have

$$2\cos^2\frac{A}{2} = 1 + \cos A$$

$$= 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c} \qquad \text{[Art. 26 (2)]}.$$

$$= \frac{\cos a - \cos(b+c)}{\sin b \sin c} = \frac{2\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}{\sin b \sin c}$$

$$= \frac{2\sin s \sin(s-a)}{\sin b \sin c}, \quad \text{where } 2s = a+b+c;$$

hence

$$\cos\frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.$$

Similarly,

$$\cos\frac{B}{2} = \sqrt{\frac{\sin s \sin(s-b)}{\sin c \sin a}}, \qquad (5)$$

nd

$$\cos\frac{C}{2} = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}.$$

Again,

$$2\sin^2\frac{A}{2} = 1 - \cos A$$

$$= 1 - \frac{\cos a - \cos b\cos c}{\sin b\sin c}$$

$$= \frac{\cos(b-c) - \cos a}{\sin b\sin c}$$

$$= \frac{2\sin(s-b)\sin(s-c)}{\sin b\sin c};$$

hence

$$\left.\begin{aligned} \sin\frac{A}{2} &= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}. \\ \text{Similarly,}\quad \sin\frac{B}{2} &= \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}}, \\ \text{and}\quad \sin\frac{C}{2} &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}. \end{aligned}\right\} \qquad (6)$$

From the above results we obtain

$$\tan\frac{A}{2} = \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}},$$

with similar expressions for $\tan\frac{B}{2}$ and $\tan\frac{C}{2}$. (7)

NOTE.—

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2}{\sin b\sin c}\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)};$$

[From (5) and (6)].

hence

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{2}{\sin a \sin b \sin c}\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}.$$

Comparing this expression for $\sin A$ with that derived in Art. 30, we see that

$$\begin{aligned}&\sin s \,.\, \sin(s-a)\,.\,\sin(s-b)\,.\,\sin(s-c)\\ &\equiv \tfrac{1}{4}(1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c)\\ &\equiv n^2,\end{aligned}$$

an *identity* which can be readily proved by direct multiplication of the four factors, $\sin s$, $\sin(s-a)$, $\sin(s-b)$, and $\sin(s-c)$.

Thus,

$$2\sin s\sin(s-a) = \cos a - \cos(b+c), \text{ and } 2\sin(s-b)\sin(s-c) = \cos(b-c) - \cos a;$$

therefore, &c.

Examples.

1. Find the conditions that

$$1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c = 0.$$

2. Prove that

$$8n^3 = \sin^2 a\sin^2 b\sin^2 c\sin A\sin B\sin C.$$

3. Prove that

$$\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = \sin(s-a) : \sin(s-b) : \sin(s-c).$$

4. Prove that

$$\tan\frac{B}{2}\tan\frac{C}{2} = \frac{\sin(s-a)}{\sin s}.$$

5. Prove that

$$\sin(s-a) = \frac{\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A}\sin a,$$

with similar expressions for $\sin(s-b)$ and $\sin(s-c)$.

6. Prove that

$$\sin s = \frac{\cos \frac{1}{2} B \cos \frac{1}{2} C}{\sin \frac{1}{2} A} \sin a.$$

7. Prove that

$$\sin a \sin b \sin c \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{n^2}{\sin s}.$$

8. Prove that

$$\sin \frac{A}{2} \cos \frac{B}{2} = \frac{\sin (s-b)}{\sin c} \cos \frac{C}{2}.$$

NOTE.*—By the aid of a similar equation,

$$\cos \frac{A}{2} \sin \frac{B}{2} = \frac{\sin (s-a)}{\sin c} \cos \frac{C}{2}.$$

We have, on adding and subtracting,

$$\sin \frac{A+B}{2} \cos \frac{c}{2} = \cos \frac{a-b}{2} \cos \frac{C}{2}. \quad (\alpha)$$

$$\sin \frac{A-B}{2} \sin \frac{c}{2} = \sin \frac{a-b}{2} \cos \frac{C}{2}. \quad (\beta)$$

Again, from Examples 5 and 6, we have

$$\cos \frac{A+B}{2} \cos \frac{c}{2} = \cos \frac{a+b}{2} \sin \frac{C}{2}. \quad (\gamma)$$

$$\cos \frac{A-B}{2} \sin \frac{c}{2} = \sin \frac{a+b}{2} \sin \frac{C}{2}, \quad (\delta)$$

four equations which will be afterwards proved geometrically.

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* The equations α, β, γ, δ, are known as Gauss's Formulæ, and the equations α', β', γ', δ' (Ex. 9), are known as Napier's Analogies. They will all be further discussed later on.

9. Prove that—

$$(\alpha'). \quad \tan\frac{A+B}{2} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

$$(\beta'). \quad \tan\frac{A-B}{2} = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

$$(\gamma'). \quad \tan\frac{a+b}{2} = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

$$(\delta'). \quad \tan\frac{a-b}{2} = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

[These results follow from Ex. 8.]

Supplemental Theorems.

34. *Having given the angles of a spherical triangle, to determine the cosines of the sides* (cf. Art. 26).

In the formula,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

change the sides into the supplements of the corresponding angles, and the angle into the supplement of the corresponding side (Note, Art. 22), and we get

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a.$$

Therefore,

Similarly,

and

$$\left.\begin{aligned} \cos a &= \frac{\cos A + \cos B \cos C}{\sin B \sin C}. \\ \cos b &= \frac{\cos B + \cos C \cos A}{\sin C \sin A}, \\ \cos c &= \frac{\cos C + \cos A \cos B}{\sin A \sin B}. \end{aligned}\right\} \qquad (8)$$

All the results obtained in Art. 26, as particular cases of formula (1) of this Chapter, may also be deduced as particular cases of formula (8), by supposing the triangle to be equilateral or right-angled, as the case may require.

35. **Analogous Formula in Plano.**—If we suppose the radius of the sphere to be indefinitely great, we have (Chap. II., Art. 18) $\cos a = 1$.

Therefore,

$$\cos A + \cos B \cos C = \sin B \sin C,$$

from which it follows at once that

$$A + B + C = \pi.$$

Examples.

1. In any triangle show that

$$\frac{\cos A + \cos B}{1 - \cos C} = \frac{\sin (a+b)}{\sin c}.$$

[If the internal bisector of the angle C make an angle θ with the opposit side, we have

$$\cos \theta + \cos A \cos \frac{C}{2} = \sin A \sin \frac{C}{2} \cos b; \text{ by Art. 34 (8)};$$

and also

$$-\cos \theta + \cos B \cos \frac{C}{2} = \sin B \sin \frac{C}{2} \cos a.$$

Adding these equations, we eliminate θ, and deduce the above expression.]

2. In any triangle show that

$$\Sigma \frac{\cos A + \cos B}{1 - \cos C} \sin (a - b) \sin c = 0.$$

[This result may be obtained at once from Ex. 1.]

3. Prove the relation

$$\frac{\cos A - \cos B}{1 + \cos C} = \frac{\sin (a \sim b)}{\sin c}.$$

[This follows, as in Ex. 1, by drawing the external, instead of the internal, bisector of C.]

4. Prove the relation

$$\Sigma \frac{\cos A - \cos B}{1 + \cos C} \sin (a + b) \sin c = 0.$$

5. If a great circle passes through the vertex C, making angles α and β with the sides a and b, the angle θ, which it makes with the side c, is given by the equation

$$\cos A \sin \alpha - \cos B \sin \beta = \pm \cos \theta \sin C.$$

[For

$$\cos A + \cos \theta \cos \beta = \sin \theta \sin \beta \cos \delta \text{ (Art. 34 (8))},$$

where δ is the intercept between the vertex and the base; also

$$\cos B - \cos \theta \cos \alpha = \sin \theta \sin \alpha \cos \delta,$$

eliminate δ between these equations; therefore, &c.]

6. If through any point P on a sphere three great circles be drawn, cutting the sides of a triangle at angles X, Y, Z; X_1, Y_1, Z_1; X_2, Y_2, Z_2, respectively; prove the following *determinant** relation :—

$$\begin{vmatrix} \cos X & \cos Y & \cos Z \\ \cos X_1 & \cos Y_1 & \cos Z_1 \\ \cos X_2 & \cos Y_2 & \cos Z_2 \end{vmatrix} = 0.$$

[Let the three concurrent arcs make angles α, β, and $\alpha + \beta$ with each other.

Since the side a is cut at angles X, X_1, and X_2; by Ex. 5,

$$\cos X \sin \alpha - \cos X_2 \sin \beta = \cos X_1 \sin (\alpha + \beta).$$

Similarly,

$$\cos Y \sin \alpha - \cos Y_2 \sin \beta = \cos Y_1 \sin (\alpha + \beta),$$

and

$$\cos Z \sin \alpha - \cos Z_2 \sin \beta = \cos Z_1 \sin (\alpha + \beta).$$

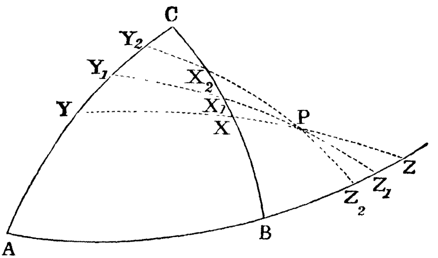

Fig. 17.

Eliminating α and β from the three equations, the above result easily follows.]

* For a knowledge of Determinants, *vide* Burnside and Panton's *Theory of Equations*, Chap. xi.

7. Having given that the sides of a triangle are each $\frac{\pi}{3}$, find the sides of the supplemental triangle.

$\left[\text{Since the angles of the latter triangle are each } \frac{2\pi}{3},\ \cos a = -\frac{1}{3}\right].$

8. Find a relation connecting the angles of a triangle if one side a is a quadrant. *Ans.* $\cos A + \cos B \cos C = 0$.

9. Given A, B, C, find the angle θ which the bisector of the vertical angle makes with the base.

[If i be the length of the internal bisector, we have

$$\cos i = \frac{\cos A + \cos\theta \cos\frac{1}{2}C}{\sin\theta \sin\frac{1}{2}C} = \frac{\cos B - \cos\theta \cos\frac{1}{2}C}{\sin\theta \sin\frac{1}{2}C},$$

therefore,

$$\cos\theta = \frac{\cos A \sim \cos B}{2\cos\frac{1}{2}C}.\Bigg]$$

10. If θ' denote the angle made by the external bisector of the vertical angle with the base, show that

$$\cos\theta' = \frac{\cos A + \cos B}{2\sin\frac{1}{2}C}. \qquad \text{(Cf. Ex. 9.)}$$

11. From any three points, A, B, and C, on a great circle, secondaries AA', BB', and CC' are drawn to another great circle; prove that the *algebraic* sum

$$\sin B'C' \cos A + \sin C'A' \cos B + \sin A'B' \cos C$$

is equal to zero. (Apply Ex. 5.)

12. The extremities of the diameter AB of a small circle are joined with a point C on the circle; prove that the angles subtended at the pole by the joining arcs AC and BC are $\cos^{-1}\frac{\sin(A \sim B)}{\sin C}$. (Apply Ex. 5.)

13. Given the base c and $\frac{\cos A}{\cos B} = -\cos C$, find the locus of the vertex.

[$a = \frac{\pi}{2}$ by Ex. 8. Hence the locus is a great circle, having the vertex B for pole.]

36. Expression for the Side of a Spherical Triangle in terms of the Trigonometrical Functions of the Angles. (Cf. Art. 33.)

We have

$$2\sin^2\frac{a}{2} = 1 - \cos a = 1 - \frac{\cos A + \cos B\cos C}{\sin B\sin C} \qquad \text{[Art. 34.]}$$

$$= -\frac{\cos A + \cos(B + C)}{\sin B\sin C} = -\frac{2\cos S . \cos(S - A)}{\sin B\sin C},$$

where

$$2S = A + B + C.$$

Therefore,

$$\sin\frac{a}{2} = \sqrt{\frac{-\cos S\cos(S - A)}{\sin B\sin C}}, \qquad (9)$$

with similar values for $\sin\frac{1}{2}b$ and $\sin\frac{1}{2}c$

Again,

$$2\cos^2\frac{a}{2} = 1 + \cos a = 1 + \frac{\cos A + \cos B\cos C}{\sin B\sin C}$$

$$= \frac{\cos A + \cos(B - C)}{\sin B\sin C} = \frac{2\cos(S - B)\cos(S - C)}{\sin B\sin C};$$

therefore,

$$\cos\frac{a}{2} = \sqrt{\frac{\cos(S - B)\cos(S - C)}{\sin B\sin C}}, \qquad (10)$$

with similar values for $\cos\frac{1}{2}b$ and $\cos\frac{1}{2}c$.

Also

$$\tan\frac{a}{2} = \frac{\sin\frac{1}{2}a}{\cos\frac{1}{2}a} = \sqrt{\frac{-\cos S\cos(S - A)}{\cos(S - B)\cos(S - C)}}, \qquad (11)$$

with similar values for $\tan\frac{1}{2}b$ and $\tan\frac{1}{2}c$.

Moreover,

$$\sin a = 2 \sin \tfrac{1}{2} a \cos \tfrac{1}{2} a$$

$$= \frac{2}{\sin B \sin C} \sqrt{- \cos S \cos(S-A) \cos(S-B) \cos(S-C)}. \tag{12}$$

We shall use the symbol N to denote the radical in this expression, so that we have

$$\sin B \sin C \sin a = \sin C \sin A \sin b = \sin A \sin B \sin c = 2N.$$

Remark.—The positive sign has been given to the radicals in the formulæ for $\sin \frac{1}{2}a$, $\cos \frac{1}{2}a$, and $\tan \frac{1}{2}a$, since $\frac{1}{2}a$ is necessarily less than $\frac{1}{2}\pi$. (See also Chap. II., Art. 23.)

Examples.

1. $$4N^2 = 1 - \cos^2 A - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C.$$

2. If p, q, r be the perpendiculars from the vertices on the opposite sides, show that—

(α). $\sin a \ \sin p = \sin b \ \sin q = \sin c \ \sin r = 2n.$

(β). $\sin A \sin p = \sin B \sin q = \sin C \sin r = 2N$;

and hence

$$\frac{\sin a}{\sin A} = \text{etc.} = \frac{n}{N}$$

3. Prove that

$$n = \frac{2N^2}{\sin A \ \sin B \ \sin C}.$$

4. Prove that

$$N = \frac{2n^2}{\sin a \sin b \sin c}.$$

5. Prove that

$$2N = (\sin a \sin b \sin c \sin^2 A \sin^2 B \sin^2 C)^{\frac{1}{3}}.$$

6. Prove that

$$\tan\frac{b}{2}\tan\frac{c}{2} = \frac{-\cos S}{\cos(S-A)}.$$

[Geometrically thus (see Chap. II., fig. 10):

$$\frac{-\cos S}{\cos(S-A)} = \frac{\tan AL}{\tan AD} : \frac{\tan AL}{\tan AY} = \cot AD \tan AY = \tan\frac{b}{2}\tan\frac{c}{2}$$

7 Prove that

$$\frac{\sin^2 A + \sin^2 B + \sin^2 C}{\sin^2 a + \sin^2 b + \sin^2 c} = \frac{1 + \cos A \cos B \cos C}{1 - \cos a \cos b \cos c}.$$

—(*Dublin Univ. Exam. Papers.*)

[We have

$$\frac{N^2}{n^2} = \frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 A + \sin^2 B + \sin^2 C}{\sin^2 a + \sin^2 b + \sin^2 c} = \frac{4N^2 - \sin^2 A - \sin^2 B - \sin^2 C}{4n^2 - \sin^2 a - \sin^2 b - \sin^2 c};$$

therefore, &c. (see Exs. 1 and 2).]

8. Given the vertical angle C of a triangle, and the function N, find the *envelope** of the base.

[By formula (12)

$$2N = \sin B \sin C \sin a = \sin C \sin r,$$

where r is the perpendicular from the vertex on the base. The envelope is, therefore, a small circle round C as pole.]

9. If $A = a$, then B and C are respectively equal or supplemental to b and c, and $N = n$.

10. Show that

$$4Nn = \sin a \sin b \sin c \sin A \sin B \sin C.$$

11. Verify the following formulæ by the expansions of $\tan\frac{1}{2}(a \pm b)$:—

$$\alpha^\circ. \quad \tan\frac{a+b}{2} = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

$$\beta^\circ. \quad \tan\frac{a-b}{2} = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

---

* The envelope is the figure which it always touches.

12. Write out and otherwise establish the corresponding formulæ to (α° and β°, Ex. 11) for the supplemental triangle.

13. If l be the length of the arc joining the middle point of the base to the vertex, find an expression for its length in terms of the sides.

[Let the arc l make an angle θ with the base c. Then, Art. 26,

$$\cos\theta = \frac{\cos a - \cos l \cos \frac{1}{2} c}{\sin l \sin \frac{1}{2} c} = -\frac{\cos b - \cos l \cos \frac{1}{2} c}{\sin l \sin \frac{1}{2} c};$$

therefore,

$$\cos l = \frac{\cos a + \cos b}{2 \cos \frac{1}{2} c}.\Big]$$

14. What is the corresponding theorem *in plano* to Ex. 13?

[Let α, β, γ, and λ be the *lengths* of the arcs a, b, c, and l, respectively. Expanding by Chap. II., Art. 18, and neglecting the powers of the radius higher than the second, we have

$$1 - \frac{\lambda^2}{2r^2} = \frac{\left(1 - \frac{\alpha^2}{2r^2}\right)\left(1 - \frac{\beta^2}{2r^2}\right)}{2\left(1 - \frac{\gamma^2}{8r^2}\right)}.$$

Hence, by reduction,

$$\alpha^2 + \beta^2 = 2\left(\frac{\gamma}{2}\right)^2 + 2\lambda^2,$$

a result well known in Geometry.]

15. Having given the base, and the sum of the cosines of the sides, find the locus of the vertex. (Cf. Ex. 13.)

16. An equilateral triangle is described on a sphere whose radius is r, and each of its sides is less than a third of the circumference of a great circle by a small difference, k. Show that the square of a side of the polar triangle is $4rk\sqrt{3}$ very nearly

SECTION III.

Three Important Theorems.

37. **Theorem I.**—In any spherical triangle,

$$\cot a \sin b = \cot A \sin C + \cos b \cos C.^*$$

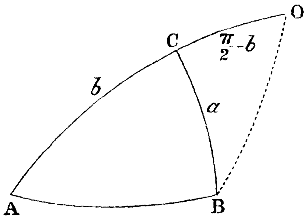

Fig. 18.

Let ABC (fig. 18) be a spherical triangle. Produce AC through C to O, so that $AO = \frac{\pi}{2}$. Join BO.

Applying Art. 26 (1), to the triangles BOC and BOA, we have

$$\cos BO = \cos a \sin b - \sin a \cos b \cos C$$

from triangle BOC; and

$$\cos BO = \sin c \cos A$$

from triangle BOA, since $\cos AO = 0$.

---

* The following symmetrical form of writing this equation, in which it is easily remembered, has been in use for some time, and is worthy of notice. Taking the four parts of the triangle connected by the equation in the order in which they occur in the triangle (e.g. a, C, b, A), and calling the first and fourth the extreme, and the second and third the middle terms, the formula becomes

$$\cos(\text{mid. side}) \cos(\text{mid. angle}) = \sin(\text{mid. side}) \cot(\text{other side}) - \sin(\text{mid. angle}) \cot(\text{other angle}).$$

Hence, equating these expressions for $\cos BO$,

$$\cos a \sin b = \sin a \cos b \cos C + \sin c \cos A.$$

Dividing each side of this equation by $\sin a$, and substituting $\frac{\sin C}{\sin A}$ for $\frac{\sin c}{\sin a}$, and we have

$$\cot a \sin b = \cot A \sin C + \cos b \cos C. \qquad (1)$$

Similarly, by producing the side BC through C to a point $\frac{\pi}{2}$ from B, we have

$$\cot b \sin a = \cot B \sin C + \cos a \cos C. \qquad (2)$$

Similarly, by producing the sides of the triangle through the angles A and B, it follows that

$$\cot b \sin c = \cot B \sin A + \cos c \cos A. \qquad (3)$$

$$\cot c \sin b = \cot C \sin A + \cos b \cos A. \qquad (4)$$

$$\cot c \sin a = \cot C \sin B + \cos a \cos B. \qquad (5)$$

$$\cot a \sin c = \cot A \sin B + \cos c \cos B. \qquad (6)$$

The same results will be obtained directly from the equations

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

. . . .

by eliminating the sides a or c.

The above formulæ, as they enable us, when given two sides and an angle, or two angles and a side of a triangle, to determine the remaining parts, are of great importance, and are thus expressed in general terms:—

Cotangent of any side × sine of another side = cotangent of the angle opposite to the first side × sine of included angle + cosine of second side × cosine of included angle.

38. Applying equation (1) of preceding Article to the polar triangle, we have

$$-\cot A \sin B = -\cot a \sin c + \cos B \cos c;$$

or, by transposing,

$$\cot a \sin c = \cot A \sin B + \cos c \cos B,$$

an equation identical with (6), Art. 37.

Moreover, it will be seen that each of the formulæ just proved contains four parts of a spherical triangle, which are connected thus :—

(α) Two sides and included angle with one of the base angles; or,

(β) Two angles and an adjacent side with one of the remaining sides.

39. **Corresponding Theorem in Plane.**—Using equation (1), and putting

$$\frac{r}{a} = \cot a, \quad \frac{\beta}{r} = \sin b, \quad \text{and } \cos b = 1,$$

we have

$$\frac{\beta}{a} = \cot A \sin C + \cos C \equiv \frac{\sin (A + C)}{\sin A}.$$

Hence

$$\frac{a}{\beta} = \frac{\sin A}{\sin B}.$$

(*Science and Art Math. Honors Exam.*)

Examples.

1. Having given two sides of a spherical triangle, find the length of the bisector of the vertical angle intercepted between the vertex and the base

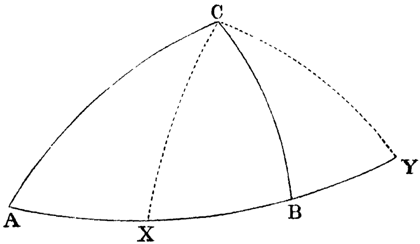

Fig. 19.

Let ι denote the internal bisector of the angle C, and θ (fig. 19) the angle it makes with the base.

Then

$$\cot a \sin \iota = \cot \theta \sin \tfrac{1}{2} C + \cos \iota \cos \tfrac{1}{2} C,$$

and

$$\cot b \sin \iota = - \cot \theta \sin \tfrac{1}{2} C + \cos \iota \cos \tfrac{1}{2} C;$$

by addition,

$$(\cot a + \cot b) \sin \iota = 2 \cos \iota \cos \tfrac{1}{2} C;$$

or

$$\cot \iota = \frac{\cot a + \cot b}{2 \cos \frac{1}{2} C}.$$

Similarly, if η denote the external bisector of C (by the colunar triangle)

$$\cot \eta = \frac{\cot a \sim \cot b}{2 \sin \frac{1}{2} C}.$$

2. What are the analogous formulæ *in plano*?

Ans. $\dfrac{1}{\text{Internal Bisector}} = \dfrac{\frac{1}{a} + \frac{1}{b}}{2 \cos \frac{1}{2} C}$, or Internal Bisector $= \dfrac{2ab}{a+b} \cos \dfrac{C}{2}$.

and External Bisector $= \dfrac{2ab}{a \sim b} \sin \dfrac{C}{2}$.

3. If ι_1, ι_2, and ι_3 denote the bisectors of the internal angles of a triangle, prove that

$$\cot \iota_1 \cos \frac{A}{2} + \cot \iota_2 \cos \frac{B}{2} + \cot \iota_3 \cos \frac{C}{2} = \cot a + \cot b + \cot c;$$

and hence, in a plane triangle (*vide* Hudson's *Plane Trig.*, p. 154),

$$\frac{\cos \frac{1}{2} A}{\iota_1} + \frac{\cos \frac{1}{2} B}{\iota_2} + \frac{\cos \frac{1}{2} C}{\iota_3} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c};$$

and if η_1, η_2, and η_3 denote the bisectors of the external angles,

$$\cot \eta_1 \sin \frac{A}{2} + \cot \eta_2 \sin \frac{B}{2} + \cot \eta_3 \sin \frac{C}{2} = 0.$$

4. Using the preceding notation, prove that

$$\cot \eta_1 \sin \frac{A}{2} = \cot \iota_2 \cos \frac{B}{2} \sim \cot \iota_3 \cos \frac{C}{2}.$$

5. Find the angles θ and θ' (fig. 19) made by the bisectors of the angle C with the opposite side c.

[From the equations of Example 1 we have at once

$$\cot \theta = \frac{(\cot a - \cot b)}{2 \sin \frac{1}{2} C} \sin \iota,$$

and

$$\cot \theta' = \frac{(\cot a + \cot b)}{2 \cos \frac{1}{2} C} \sin \eta.]$$

6. Find the arc intercepted on the base by the bisectors of the vertical angle

[$\cos XY = \cot \theta \cot \theta'$ (fig. 19).

$$= \frac{(\cot^2 a - \cot^2 b)}{2 \sin C} \sin \iota \sin \eta; \text{ (Ex. 5)}$$

but $\sin \iota \sin \eta = \sin XY \sin r$ (Art. 31, Ex. 2), where r is the perpendicular from the vertex C on the base AB.

Therefore

$$\cot XY = \frac{(\cot^2 a - \cot^2 b) \sin a \sin b}{2 \sin c} \text{ (Art. 31, Exs. 1 and 2.)}$$

$$= \frac{\sin (a+b) \sin (a-b)}{2 \sin a \sin b \sin c};$$

or, in terms of the angles,

$$\cot XY = \frac{\sin^2 A - \sin^2 B}{2 \sin A \sin B \sin c}.]$$

7. Show that the sum of the cotangents of the intercepts made by the internal and external bisectors of the angles of a spherical triangle on the opposite sides is equal to zero. (*Educational Times.*)

[By the preceding Example

$$\Sigma \cot XY = \Sigma \frac{\sin^2 a - \sin^2 b}{2 \sin a \sin b \sin c} = 0.]$$

8. Give the analogous property *in plano*.

(See Casey's *Sequel to Euclid*, sec. viii., Ex. 17.)

9. If ι_1, ι_2, and ι_3, the internal bisectors of the angles of a triangle, make angles respectively θ_1, θ_2, θ_3, with the opposite sides, prove that

$$\frac{\sin \frac{1}{2} A}{\sin \iota_1 \tan \theta_1} + \frac{\sin \frac{1}{2} B}{\sin \iota_2 \tan \theta_2} + \frac{\sin \frac{1}{2} C}{\sin \iota_3 \tan \theta_3} = 0;$$

(See Ex. 5.)

and that if θ_1', θ_2', and θ_3', are the angles made by the external bisectors with the opposite sides,

$$\frac{\cos \frac{1}{2} A}{\sin \eta_1 \tan \theta'_1} + \frac{\cos \frac{1}{2} B}{\sin \eta_2 \tan \theta'_2} + \frac{\cos \frac{1}{2} C}{\sin \eta_3 \tan \theta'_3} = \cot a + \cot b + \cot c.$$

(See Ex. 5.)

10. From the vertex C of a spherical triangle an arc of a great circle is drawn to meet the base AB in D; if C_1 and C_2 are the segments of the angle C, respectively, opposite AD and BD, show that

$$\cot CD \sin C = \cot a \sin C_1 + \cot b \sin C_2 *.$$

(*Science and Art Math. Honors.*)

[Let CD make an angle θ with the base, and we have

$$\cot a \sin CD = \cot \theta \sin C_2 + \cos CD \cos C_2 \qquad (1)$$

also,

$$\cot b \sin CD = -\cot \theta \sin C_1 + \cos CD \cos C_1 \qquad (2)$$

Eliminate θ from equations (1) and (2); therefore, &c.]

11. If ϕ be the angle between the base c of a spherical triangle and its bisector,

$$\cot A - \cot B = 2 \cos \frac{c}{2} \cot \phi.$$

---

* This relation connects the latitudes and *differences* of longitudes of three places in the same great circle on the surface of the Earth.

12. Prove that

$$\frac{\cos^2 b - \cos^2 c}{\cos b \cot B - \cos c \cot C} = \frac{\cos^2 c - \cos^2 a}{\cos c \cot C - \cos a \cot A}$$
$$= \frac{\cos^2 a - \cos^2 b}{\cos a \cot A - \cos b \cot B},$$

and express each of the three quantities as a symmetric function of the sides of the triangle.

[Multiplying equation (1) by $\cos a$ and (2) by $\cos b$, and subtracting the results, each of the expressions becomes $\sin b \sin c \sin A$, &c. $= 2n$.]

13. Write out the corresponding theorem for the supplemental triangle.

40. **Theorem II.**—*The angular distances of three points, A, B, and C, on the same great circle, and any other point, P, on the sphere, are connected by the relation*

$$\sin BC \cos AP + \sin CA \cos BP + \sin AB \cos CP = 0.$$

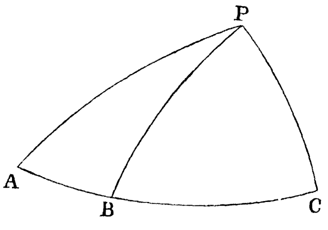

Fig. 20.

We have (fig. 20),*

$$\cos AP = \cos AB \cos BP + \sin AB \sin BP \cos ABP, \quad (1)$$

and

$$\cos CP = \cos BC \cos BP - \sin BC \sin BP \cos ABP. \quad (2)$$

* Or thus:—If D be the foot of the perpendicular p from P on AC, then $\Sigma \sin BC \cos AD = 0$. But $\cos AD = \cos AP \div \cos p$; therefore, &c.

Multiply—(1) by sin BC, and (2) by sin AB, and add the results. Then

$$\sin BC \cos AP + \sin AB \cos CP = \sin AC \cos BP;$$

or writing $\sin AC = -\sin CA$, we have the algebraic sum

$$\sin BC \cos AP + \sin CA \cos BP + \sin AB \cos CP$$

equal to zero.*

41. **Analogous Theorem in Plano.**—By the method similar to that used in Art. 39, we get

$$BC.AP^2 + CA.BP^2 + AB.CP^2 = -BC.CA.AB.$$

(Townsend's *Modern Geometry*, vol. i.)

42. When the point B (fig. 20) bisects the base AC of the triangle ACP, the formula of Art. 40 readily reduces to

$$\cos AP + \cos CP = 2 \cos \tfrac{1}{2} AC \cos BP.$$

Or,

The sum of the cosines of the sides of a triangle is equal to twice the cosine of half the base into cosine of the bisector of the base. [See also Ex. 13, Art. 36.]

Again, when the point B on the base AC is ninety degrees distant from the middle point of AC, we have

$$\cos AP - \cos CP = 2 \sin \frac{AC}{2} \cos BP,$$

an equation which furnishes an obvious solution to the problem—*Given the base and difference of cosines of the sides, find the locus of the vertex.*

---

* Compare this with the more general results of Arts. 132 and 134.

Examples.

1. Having given the base c of a triangle, and $l \cos a \pm m \cos b$, find the locus of the vertex, l and m being constants.

Ans. A circle, the pole of which divides the base internally (or externally) into segments, the sines of which are to one another as l is to m.

NOTE.—Particular cases of this example, viz., when $l = \pm m$, have been considered in Arts. 42 and 36.

2. Having given the base and ratio of cosines of the sides; find the locus of the vertex.

[Let the arc CP (fig. 20) be a quadrant. Then in the triangle ABP, the formula of Art. 40 reduces thus:

$$\frac{\cos AP}{\cos BP} = \frac{\sin AC}{\sin BC};$$

hence, according to the given conditions, C is a fixed point on the base, and the vertex therefore describes a great circle having C for pole. (Cf. Art. 56, Ex. 11.)]

3. Show that the points A, B, C (fig. 20), will lie on the same great circle if

$$\sin BC \sin AL + \sin CA \sin BL + \sin AB \sin CL = 0,$$

where AL, BL, CL, are secondaries to any great circle L.

What is the corresponding theorem *in plano*?

(*Vide* Townsend's *Modern Geometry*, vol. i., chap. 5.)

4. If the base c of a triangle be a quadrant, show that if any point X be taken on it,

$$\cos CX = \cos a \sin AX + \cos b \cos AX.$$

5. From any three points on a great circle, secondaries, x, y, z; x', y', z'; x'', y'', z'', are drawn to the sides of a triangle; prove the determinant relation

$$\begin{vmatrix} \sin x & \sin y & \sin z \\ \sin x' & \sin y' & \sin z' \\ \sin x'' & \sin y'' & \sin z'' \end{vmatrix} = 0.$$

(Apply Ex. 3.)

43. **Theorem III.**—*If from a fixed point P on the surface of a sphere an arc of a great circle be drawn, meeting a small circle at A and B, and if PA and PB be denoted by ρ and ρ', then*

$$\tan \tfrac{1}{2}\rho \tan \tfrac{1}{2}\rho' = \text{constant}.$$

Let C (fig. 21) be the pole of the small circle. Join CA, CB, and CP by arcs of great circles. Let the angle

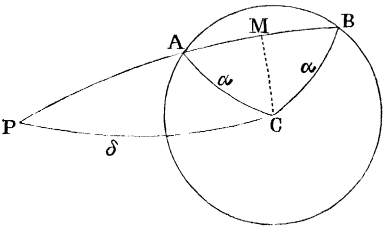

Fig. 21.

APC be denoted by θ, the arc CP by δ, and the arc CA by a.

From the triangle APC we have

$$\cos a = \cos\rho \cos\delta + \sin\rho \sin\delta \cos\theta.$$

Let $\tan \frac{1}{2}\rho = t$. Then $\cos\rho = \dfrac{1 - t^2}{1 + t^2}$, and $\sin\rho = \dfrac{2t}{1 + t^2}$.

Substituting these values for $\cos\rho$ and $\sin\rho$ in the above equation, and arranging in powers of t, we get

$$t^2(\cos a + \cos\delta) - 2t \sin\delta \cos\theta + \cos a - \cos\delta^* = 0. \quad (1)$$

* This may be regarded as the polar equation of the small circle, and its analogue *in plano*, $\rho^2 - 2\rho d \cos\theta + d^2 - a^2 = 0$, can be deduced at once by aid of the formulæ of Art. 18. If for t we substitute its reciprocal $\frac{1}{t}$, we form a new equation, the roots of which are the reciprocals of the roots of equation (1). The product of the roots of the reciprocal equation is clearly constant, and it is therefore the equation of a circle. It therefore follows that if on any radius vector, PA, drawn from any fixed point P to a circle, a point A' be taken, such that $\tan \frac{1}{2} PA \tan \frac{1}{2} PA' =$ constant, then the locus of A' is also a circle. This method of transformation is termed *Spherical Inversion*, and a summary of it will be found in Chapter XII., Section v.

By proceeding in identically the same manner we obtain from the triangle BPC an equation in t' (where $t' = \tan \frac{1}{2} \rho'$) in all respects the same as that in t', and hence it follows that $\tan \frac{1}{2} \rho$ and $\tan \frac{1}{2} \rho'$ are the roots of the above quadratic, and therefore their product,

$$\tan \tfrac{1}{2} \rho \tan \tfrac{1}{2} \rho' = \frac{\cos a - \cos \delta}{\cos a + \cos \delta} = \tan \tfrac{1}{2} (\delta - a) \tan \tfrac{1}{2} (\delta + a), \quad (2)$$

which value is independent of θ, and depends only on the fixed quantities a and δ; therefore, &c.

Otherwise thus :—

Draw CM perpendicular to AB. Then $AM = BM$; and since (Art. 27, Ex. 1) $\cos \delta = \cos PM \cos CM$, and $\cos a = \cos AM \cos CM$, we have, by division,

$$\frac{\cos AM}{\cos PM} = \frac{\cos a}{\cos \delta}.$$

Therefore,

$$\frac{\cos AM - \cos PM}{\cos AM + \cos PM} = \frac{\cos a - \cos \delta}{\cos a + \cos \delta},$$

which reduces at once to

$$\tan \tfrac{1}{2} \rho \tan \tfrac{1}{2} \rho' = \tan \tfrac{1}{2} (\delta - a) \tan \tfrac{1}{2} (\delta + a).$$

Remark.—In fig. 21 it is to be observed that we have *two* triangles, CPA and CPB, having the same *three* parts a, δ, and θ. Whenever more than one triangle can be constructed from three given parts, the solution of the triangle is said to be ambiguous (see Chap. V.) However, it should be noted that the remaining sides ρ and ρ' of the triangles are not independent of each other, but are connected by the relation of this Article.

Examples.

1. Deduce the *corresponding theorem in plano.*

Ans.—Euclid, III. XXXVI.

2. If the vertical angle C of a triangle be equal to the sum of the base angles, and if the perpendicular p from the vertex on the base divides it into segments c_1 and c_2, show that

$$\tan \tfrac{1}{2} c_1 \tan \tfrac{1}{2} c_2 = \tan^2 \tfrac{1}{2} p.$$

[Let p meet the base in D. Produce CD through D to C', so that $DC' = DC$. Then the triangles ABC and ABC' are equal in all respects, and the vertices of the quadrilateral $ACBC'$ are concyclic, being equidistant from the middle point of AB. Therefore, by Theorem III.,

$$\tan \tfrac{1}{2} c_1 \tan \tfrac{1}{2} c_2 = \tan \tfrac{1}{2} CD \tan \tfrac{1}{2} C'D = \tan^2 \tfrac{1}{2} p.]$$

3. Given the base AB, and that $C = A + B$, find the locus of the vertex C.

Ans.—A circle having its pole at the middle point of the base.

[Note.—In this case the chordal triangle is right-angled at C.]

4. In Theorem III., fig. 21, show that $\tan \tfrac{1}{2} \rho + \tan \tfrac{1}{2} \rho'$ is a maximum, when PAB passes through C, and determine the maximum value. (Use the quadratic in t.)

Ans. $\dfrac{2 \sin \delta}{\cos \alpha + \cos \delta}$.

5. Show that $\dfrac{\sin \frac{1}{2}(\rho + \rho')}{\sin \frac{1}{2} \rho \sin \frac{1}{2} \rho'}$, varies as $\cos \theta$, and explain when $\theta = 90°$.

6. If through a point inside a circle two chords be drawn at right angles, and if their segments be ρ_1, ρ_2; ρ_3, ρ_4, show that

$$\tan^2 \tfrac{1}{2} \rho_1 + \tan^2 \tfrac{1}{2} \rho_2 + \tan^2 \tfrac{1}{2} \rho_3 + \tan^2 \tfrac{1}{2} \rho_4 = \text{const.} = \frac{4 \sin^2 \alpha}{(\cos \alpha + \cos \delta)^2}.$$

(*Dub. Univ. Exam. Papers.*)

[Equation (1) gives us

$$\tan \tfrac{1}{2} \rho_1 + \tan \tfrac{1}{2} \rho_2 = \frac{2 \sin \delta \cos \theta}{\cos \alpha + \cos \delta},$$

and for the perpendicular chord

$$\tan \tfrac{1}{2} \rho_3 + \tan \tfrac{1}{2} \rho_4 = \frac{2 \sin \delta \sin \theta}{\cos \alpha + \cos \delta}.$$

Square, add, and reduce by equation (2).]

7. Using the notation of Ex. 6, show that

$$\cot^2 \tfrac{1}{2} \rho_1 + \cot^2 \tfrac{1}{2} \rho_2 + \cot^2 \tfrac{1}{2} \rho_3 + \cot^2 \tfrac{1}{2} \rho_4 = \frac{4 \sin^2 \alpha}{(\cos \alpha - \cos \delta)^2}.$$

8. Describe a circle bisecting the circumferences of three small circles.

TABLE OF FORMULAE.

FORMULAE.	SUPPLEMENTAL FORMULAE.	ANALOGOUS FORMULAE IN PLANO.
$\cos a = \cos b \cos c + \sin b \sin c \cos A$. (Art. 26.) $\cos b = \cos c \cos a + \sin c \sin a \cos B$. ,, $\cos c = \cos a \cos b + \sin a \sin b \cos C$. ,,	$\cos A + \cos B \cos C = \sin B \sin C \cos a$. (Art. 34.) $\cos B + \cos C \cos A = \sin C \sin A \cos b$. ,, $\cos C + \cos A \cos B = \sin A \sin B \cos c$. ,,	$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, (Art. 28.) and $A + B + C = \pi$. (Art. 35.)
$\sin \frac{1}{2}A = \sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$. (Art. 33. $\cos \frac{1}{2}A = \sqrt{\dfrac{\sin s \sin(s-a)}{\sin b \sin c}}$. ,, $\tan \frac{1}{2}A = \sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}$. ,, With similar expressions for $\sin \frac{1}{2}B$, etc., and $\sin \frac{1}{2}C$, etc.	$\sin \frac{1}{2}a = \sqrt{-\dfrac{\cos S \cos(S-A)}{\sin B \sin C}}$. (Art. 36.) $\cos \frac{1}{2}a = \sqrt{\dfrac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}$. ,, $\tan \frac{1}{2}a = \sqrt{-\dfrac{\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}}$. ,,	$\sin \frac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$. $\cos \frac{1}{2}A = \sqrt{\dfrac{s(s-a)}{bc}}$. $\tan \frac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$. and $A + B + C = \pi$.
$\sin A = \dfrac{2n}{\sin b \sin c}$. (Art. 30.) $2n = 2\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}$. $= \sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a \cos b \cos c}$.	$\sin a = \dfrac{2N}{\sin B \sin C}$. (Art. 36.) $2N = 2\sqrt{-\cos S \cos(S-A)\cos(S-B)\cos(S-C)}$ $= \sqrt{1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A \cos B \cos C}$.	$\sin A = \dfrac{2\Delta}{bc}$. $2\Delta = 2\sqrt{s(s-a)(s-b)(s-c)}$. $A + B + C = \pi$.
$\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c} = \dfrac{2n}{\sin a \sin b \sin c}$. (Art. 31.)	$\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C} = \dfrac{2N}{\sin A \sin B \sin C}$ (Art. 31.)	$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{2\Delta}{abc} = \dfrac{1}{D}$, Δ being the area of the triangle, and D the diameter of its circumscribed circle.
$\cot a \sin b = \cot A \sin C + \cos b \cos C$. (Art. 37.) $\cot b \sin c = \cot B \sin A + \cos c \cos A$. ,, $\cot c \sin a = \cot C \sin B + \cos a \cos B$. ,,	$\cot a \sin c = \cot A \sin B + \cos c \cos B$. (Art. 38.) $\cot b \sin a = \cot B \sin C + \cos a \cos C$. ,, $\cot c \sin b = \cot C \sin A + \cos b \cos A$. ,,	$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = $ etc. (Art. 39.)

Miscellaneous Examples.

1 If s and s' are the segments of the base made by the perpendicular from the vertex, and σ and σ' those made by the bisector of the vertical angle, show that

$$\tan\frac{s-s'}{2}\tan\frac{\sigma-\sigma'}{2}=\tan^2\frac{a-b}{2}.$$

—(*Dublin Univ. Exam. Papers.*)

[We have

$$\frac{\cos a}{\cos b}=\frac{\cos s}{\cos s'} \text{ and } \frac{\sin a}{\sin b}=\frac{\sin\sigma}{\sin\sigma'}.$$

Taking the difference to the sum in each equation, we get

$$\tan\frac{a-b}{2}\tan\frac{a+b}{2}=\tan\frac{s-s'}{2}\tan\frac{c}{2}, \tag{1}$$

and

$$\tan\frac{a-b}{2}\cot\frac{a+b}{2}=\tan\frac{\sigma-\sigma'}{2}\cot\frac{c}{2}. \tag{2}$$

Multiplying (1) and (2) together, we have

$$\tan\frac{s-s'}{2}\tan\frac{\sigma-\sigma'}{2}=\tan^2\frac{a-b}{2}.\ ]$$

2. Prove that

$$\sin b\sin c+\cos b\cos c\cos A=\sin B\sin C-\cos B\cos C\cos a.$$

[If from the extremities of the base BC quadrants CC' and BB' be measured on the sides towards the vertex, the left-hand side of this equation is obviously $\cos B'C'$.

Again, if secondaries be drawn to the sides at C and B, the angle between them is equal to the arc $B'C'$ (Chap. I., Art. 6), and the cosine of this angle is equal to the expression on the right-hand side of the equation by Chap. III., Art. 34; therefore, &c.]

Otherwise thus :—

Multiplying together the equations

$$\cos a=\cos b\cos c+\sin b\sin c\cos A,$$

and

$$\sin B\sin C\cos a-\cos B\cos C=\cos A,$$

and observing that

$$\sin B\sin C\sin^2 a=\sin b\sin c\sin^2 A, \qquad \text{(Art. 31.)}$$

the above result is obtained.

3. Given a, b, and c, the sides of a triangle, find γ, the arc joining the middle points of a and b.

We have

$$\cos\gamma = \cos\frac{a}{2}\cos\frac{b}{2} + \sin\frac{a}{2}\sin\frac{b}{2}\cos C$$

$$= \cos\frac{a}{2}\cos\frac{b}{2} + \frac{\cos c - \cos a\cos b}{4\cos\frac{a}{2}\cos\frac{b}{2}} \qquad \text{(by Art. 26.)}$$

$$= \frac{(1+\cos a)(1+\cos b) + \cos c - \cos a\cos b}{4\cos\frac{a}{2}\cos\frac{b}{2}}$$

$$= \frac{1+\cos a+\cos b+\cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}},$$

with similar expressions for the arcs joining the middle points of b and c, c and a.

Note.—In the colunar triangle

$$\cos\gamma' = \frac{1+\cos c-\cos a-\cos b}{4\sin\frac{a}{2}\sin\frac{b}{2}}.$$

4. If α, β, and γ be the arcs joining the middle points of the sides, prove

$$\frac{\cos\alpha}{\cos\frac{a}{2}} = \frac{\cos\beta}{\cos\frac{b}{2}} = \frac{\cos\gamma}{\cos\frac{c}{2}} = \frac{1+\cos a+\cos b+\cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}} = \sin S;$$

and hence, if one of the arcs be a quadrant, the others are also quadrants. (See Ex. 3, and fig. 38, Art. 104.)

5. If the side c of a spherical triangle be a quadrant, and δ the arc drawn at right angles to it from the opposite vertex, show that

$$\cot^2\delta = \cot^2 A + \cot^2 B.$$

—(*Science and Art Exam. Papers.*)

[Let X be the foot of the perpendicular δ on the base AB; then

$$\tan\delta = \tan A\ .\ \sin AX = \tan B\ .\ \cos AX;$$

therefore, &c.]

6. Given the vertical angle in position and magnitude, and the ratio of the cosines of the base angles, find the locus of the pole of the base.

[Let the perpendicular on the base divide the vertical angle into parts α and β; then by Art. 26, Ex. 6,

$$\frac{\cos A}{\cos B} = \frac{\sin \alpha}{\sin \beta}$$

therefore the perpendicular is fixed in direction, and it obviously contains the pole of the base; therefore, &c.]

7. Prove that in fig. 19,

$$2 \cot XY = \cot AY + \cot BY.*$$

[For

$$\frac{\sin AY}{\sin BY} = \frac{\sin AX}{\sin XB} = \frac{\sin (AY - XY)}{\sin (XY - BY)};$$

therefore, &c.]

8. If a, b, c, d be the sides of a spherical quadrilateral taken in order, and δ and δ' the diagonals; the arc ϕ joining the middle points of the diagonals is given by the equation

$$\cos \phi = \frac{\cos a + \cos b + \cos c + \cos d}{4 \cos \frac{\delta}{2} \cos \frac{\delta'}{2}},$$

and give the corresponding theorem *in plano*. (See Art. 36, Ex. 14.)

[The sum of the squares of the sides of a quadrilateral = the sum of the squares of its diagonals + four times the square of the line joining the middle points of the diagonals. (See M'Dowell's *Geom. Exercises.*)

9. The arc ϕ' joining the middle points of the opposite sides b and d of a quadrilateral is given by the equation

$$\cos \phi' = \frac{\cos a + \cos c + \cos \delta + \cos \delta'}{4 \cos \frac{b}{2} \cos \frac{d}{2}},$$

---

* An arc AB, divided internally in X and externally in Y, such that $\sin AY : \sin BY :: \sin AX : \sin XB$, is said to be cut harmonically by the segment XY; and the cotangents of either triad of arcs, viz. AY, XY, BY, or AX, AB, AY, measured from either extremity, are therefore in arithmetical progression.

with a similar expression for the arc joining the middle points of a and b. (See Ex. 8.)

10. If the arcs joining the middle points of opposite sides of a quadrilateral be each quadrants, the arc ϕ joining the middle points of the diagonals δ and δ' is given by the equation

$$-\cos\phi = \frac{\cos\delta + \cos\delta'}{2\cos\frac{\delta}{2}\cos\frac{\delta'}{2}}.$$

11. Prove that the angle ϕ between the perpendicular from the vertex on the base and the bisector of the vertical angle is thus given:

$$\tan\phi = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)} \cdot \tan\frac{A-B}{2}.^{*}$$

—(*Dub. Univ. Exam. Papers.*)

[For

$$\frac{\cos A}{\cos B} = \frac{\sin\left(\frac{C}{2}+\phi\right)}{\sin\left(\frac{C}{2}-\phi\right)};$$

therefore, taking the difference to sum,

$$\tan\frac{A+B}{2}\tan\frac{A-B}{2} = \cot\frac{C}{2}\tan\phi.$$

Now substitute the value of $\tan\frac{A+B}{2}$ in Art. 33, Ex. 9; therefore, &c.]

12. Given the base of a triangle and the bisectors of the vertical angle, construct it.

[Using the notation, fig. 19,

$$\cos XY = \cos i\cos\eta;$$

also

$$\cot AX + \cot AY = 2\cot AB, \qquad \text{(Ex. 7.)}$$

two equations which determine the points X and Y.]

---

* Tan ϕ is also equal to

$$\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\tan\frac{1}{2}(A+B).$$

13. Given the base AB and the sum of the sides of a triangle, find the locus of a point O, where the external bisector of the vertical angle C meets the secondary to BC at the point B. (Cf. Galbraith and Haughton's *Euclid*, I. II. III., Ex. 200.)

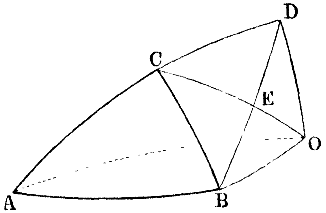

Fig. 22.

[Produce AC (fig. 22) through C to D, such that $CD = BC$. Therefore $AD =$ given sum of the sides.

Since CO bisects the angle BCD, it bisects BD at right angles.

Since BD is bisected at right angles, $BO = OD$, and the triangles CDO and CBO are equal in every respect; hence the angle CDO is a right angle. Therefore

$$\cos AD = \frac{\cos AO}{\cos OD} = \frac{\cos AO}{\cos BO}.$$

In the triangle ABO we have thus given the base AB and ratio of the cosines of the sides, to find the locus of the vertex, for a solution of which see Art. 42, Ex. 2.]

14. Given the base AB and difference of sides, find the locus of the point O, where the secondary to BC at the point B meets the internal bisector of the vertical angle. (See Ex. 13.)

15. Two places on the Earth's surface are distant, one θ° from the Pole the other θ° from the Equator, and their difference of longitude is ϕ; find the angular distance between them. —(*Science and Art Exam. Papers.*)

$$\cos \delta = \sin 2\theta \cos^2 \frac{\phi}{2}.$$

16. In an isosceles triangle, if each of the base angles be double the vertical angle, prove that

$$\cos a \cos \frac{a}{2} = \cos \left(c + \frac{a}{2}\right).$$

—(*Lond. Univ. Exam. Papers.*)

[If x denote one of the equal bisectors of the sides, we have

$$\cos x = \cos a \cos \frac{a}{2} + \sin a \sin \frac{a}{2} \cos A,$$

and

$$\cos x = \cos c \cos \frac{a}{2} + \sin c \sin \frac{a}{2} \left(2 \cos^2 A \quad 1.\right),$$

where A is the vertical angle.

Again,

$$\frac{\sin a}{\sin c} = 2 \cos A,$$

eliminating $\cos A$; therefore, &c.]

17. If a side c of a spherical triangle be a quadrant, show that

(α). $\cot a \cot b + \cos C = 0.$

(β). $\cos S \cos (S - C) + \cos (S - A) \cos (S - B) = 0.$

—(*Science and Art Exam. Papers.*)

(See Art. 26.) Also

$$\tan \frac{c}{2} = 1;$$

therefore, &c., Art. 36.

18. If λ, μ, and ν be the segments towards the angles of the internal bisectors of the angles of a triangle ABC from the point of concurrence, show that

$$\begin{vmatrix} \cos \lambda & \cos \mu & \cos \nu \\ \sin a & \sin b & \sin c \\ \cos a & \cos b & \cos c \end{vmatrix} = 0.$$

[We have

(1). $\cos \lambda = \cos b \cos \nu + \sin b \sin \nu \cos \frac{C}{2}.$

(2). $\cos \mu = \cos a \cos \nu + \sin a \sin \nu \cos \frac{C}{2}.$

Eliminate $\cos \frac{C}{2}$. Therefore,

$$\cos \lambda \sin a - \cos \mu \sin b = \cos \nu \sin (a - b)$$

Similarly,

$$\cos\mu \sin b - \cos\nu \sin c = \cos\lambda \sin(b - c),$$

and

$$\cos\nu \sin c - \cos\lambda \sin a = \cos\mu \sin(c - a);$$

whence, on adding,

$$\Sigma \cos\lambda \sin(b - c),$$

or its equivalent determinant form, given above, is equal to zero.]

19. If λ, μ, and ν be the segments towards the angles of two external bisectors (of angles B and C), and one internal bisector of the angles of a triangle from the point of concurrence, show that

$$\begin{vmatrix} -\cos\lambda & \cos\mu & \cos\nu \\ \sin a & \sin b & \sin c \\ -\cos a & \cos b & \cos c \end{vmatrix} = 0.$$

20. Using the notation of Example 18, prove that

$$\Sigma^* \sin\lambda \sin(b - c) \cos\frac{A}{2} = 0.$$

[Multiplying Ex. 18, (1) by $\cos a$, and (2) by $\cos b$, and subtracting, we get

$$\cos a \cos\lambda - \cos b \cos\mu + \sin(a - b) \sin\nu \cos\frac{C}{2} = 0,$$

which, added to similar equations, involving b and c, and c and a, gives the required result.]

21. Using the notation of Ex. 19, show that

$$\begin{vmatrix} \sin\lambda \cos\frac{A}{2} & \sin\mu \sin\frac{B}{2} & \sin\nu \cos\frac{C}{2} \\ \sin a & \sin b & \sin c \\ -\cos a & \cos b & \cos c \end{vmatrix} = 0.$$

---

* The equivalent determinant form is

$$\begin{vmatrix} \sin\lambda \cos\frac{A}{2} & \sin\mu \cos\frac{B}{2} & \sin\nu \cos\frac{C}{2} \\ \sin a & \sin b & \sin c \\ \cos a & \cos b & \cos c \end{vmatrix} = 0.$$

22. If θ, ϕ, and ψ denote the arcs connecting the middle points of pairs of opposite sides and diagonals of a quadrilateral, viz. a and b, c and d, and δ and δ' respectively, prove that

$$\cos\theta\cos\frac{a}{2}\cos\frac{b}{2}+\cos\phi\cos\frac{c}{2}\cos\frac{d}{2}-\cos\psi\cos\frac{\delta}{2}\cos\frac{\delta'}{2}=\tfrac{1}{2}(\cos\delta+\cos\delta').$$

(See Exs. 8 and 9.)

23. A, B, and C are three concyclic points on a sphere; X, Y, and Z are three other points; show that the relation

$$\begin{vmatrix} \cos AX & \cos AY & \cos AZ \\ \cos BX & \cos BY & \cos BZ \\ \cos CX & \cos CY & \cos CZ \end{vmatrix} = 0.$$

[By aid of Art. 40 we can eliminate $\sin BC$, $\sin CA$, and $\sin AB$, from three given equations; therefore, &c.]

24. If the sides of a quadrilateral taken in order are a, b, c, d, show that the diagonals δ and δ' intersect at an angle θ, given by the equation

$$\cos\theta = \frac{\cos a\cos c - \cos b\cos d}{\sin\delta\sin\delta'}.$$

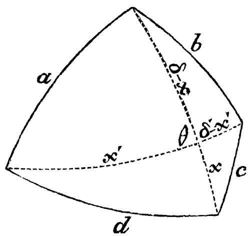

Fig. 23.

[For, representing the segments of the diagonals by x, $\delta - x$, and x', $\delta' - x'$, respectively, we have

$$\cos a = \cos x'\cos(\delta - x) + \sin x'\sin(\delta - x)\cos\theta. \quad (1)$$
$$\cos c = \cos x\cos(\delta' - x') + \sin x\sin(\delta' - x')\cos\theta. \quad (2)$$
$$\cos b = \cos(\delta - x)\cos(\delta' - x') - \sin(\delta - x)\sin(\delta' - x')\cos\theta. \quad (3)$$
$$\cos d = \cos x\cos x' - \sin x\sin x'\cos\theta. \quad (4)$$

By subtracting the product of equations (3) and (4) from the product of (1) and (2), the above expression for θ is obtained.]

25. By a method similar to that used in Ex. 24, show that if ϕ and ψ are the angles between a and c, b and d, respectively,

$$\cos\psi = \frac{\cos c \cos a - \cos\delta \cos\delta'}{\sin b \sin d}. \qquad (a)$$

$$\cos\phi = \frac{\cos b \cos d - \cos\delta \cos\delta'}{\sin c \sin a}. \qquad (b)$$

26. An arc δ drawn from the vertex of a triangle to meet the base at an angle θ may be expressed by the equation

$$\sin\delta = \frac{\cos a \cos y - \cos b \cos x}{\sin c \cos\theta},$$

where x and y are the segments of the base adjacent to the sides a and b respectively.

[In the figure of Ex. 24 suppose three vertices of the quadrilateral to be concyclic.]

27. In any triangle, prove that

$$\tan\frac{a}{2} : \tan\frac{b}{2} : \tan\frac{c}{2} = \cos(S-A) : \cos(S-B) : \cos(S-C).$$

(See Art. 36 (11.)

28. Prove the following expression* for $\cos S$:—

$$-\cos S = \frac{\sin\frac{1}{2}a \sin\frac{1}{2}b \sin C}{\cos\frac{1}{2}c} = \frac{n}{2\cos\frac{1}{2}a \cos\frac{1}{2}b \cos\frac{1}{2}c}.$$

(Apply Art. 36, (9) and (10).)

Also

$$\sin s = \frac{\cos\frac{1}{2}A \cos\frac{1}{2}B \sin c}{\sin\frac{1}{2}C}.$$

---

* The equation for calculating the sum of the angles, having given two sides and the included angle, or, by the aid of Art. 30, the three sides of a spherical triangle, will be afterwards proved as *Cagnoli's Theorem*. (Art. 103.)

29. Prove that

$$\sin(S-A)=\frac{1+\cos a-\cos b-\cos c}{4\cos\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}},$$

and similar expressions for $\sin(S-B)$ and $\sin(S-C)$.

[For

$$\sin(S-A)=\sin\frac{B+C}{2}\cos\frac{A}{2}-\cos\frac{B+C}{2}\sin\frac{A}{2};$$

therefore, &c., by the aid of note on Ex. 8, Art. 33.

Or thus:—

Take the colunar triangle, whose angles are $\pi-B$, $\pi-C$, A, and sides $\pi-b$, $\pi-c$, a (Art. 16). Let

$$2S'=\pi-B+\pi-C+A=2\pi-(B+C-A);$$

hence

$$S'=\pi-(S-A),$$

and the formula adapted to the colunar triangle becomes

$$\sin S'=\frac{1+\cos a+\cos b+\cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}};$$

therefore, &c.] (Cf. Ex. 4.)

30. Prove that

$$\cos(s-a)=\frac{1+\cos B+\cos C-\cos A}{4\cos\frac{B}{2}\cos\frac{C}{2}\sin\frac{A}{2}}.$$

31. Prove that

$$\cos(S-A)=\frac{\cos\frac{b}{2}\cos\frac{c}{2}}{\cos\frac{a}{2}}\sin A.$$

32. Prove that

$$\sin(S-A)=\frac{\sin\frac{b}{2}\sin\frac{c}{2}+\cos\frac{b}{2}\cos\frac{c}{2}\cos A}{\cos\frac{a}{2}}.$$

(See Ex. 29.)

[It is evident that the numerator of the fraction is equal to the cosine of the arc joining the middle points of the sides $\pi - b$, $\pi - c$, of the colunar triangle; therefore, &c. (Art. 26, and Ex. 4.)]

33. In any triangle,

$$\frac{\cos a - \cos b}{1 - \cos c} + \frac{\sin (A - B)}{\sin C} = 0.$$

(See Art. 35, Ex. 3.)

34. Given the formula, Art. 36, for $\tan\frac{a}{2}$ prove that

$$-\tan S = \frac{\cot\frac{a}{2}\cot\frac{b}{2} + \cos C.}{\sin C}.$$

35. Show that

$$\cot\frac{a}{2}\cos(S - A) = \cot\frac{b}{2}\cos(S - B) = \cot\frac{c}{2}\cos(S - C)$$

$$= -\cot\frac{a}{2}\cot\frac{b}{2}\cot\frac{c}{2}\cos S = \frac{n}{2\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c};$$

and hence

$$\Sigma\cos^2(S - A) = \cos^2 S\,\Sigma\cot^2\frac{a}{2}\cot^2\frac{b}{2}.$$

36. If

$$\tan\frac{a}{2}\tan\frac{b}{2} = \tan^2\frac{c}{2},$$

show that

$$\cos(S - A)\cos(S - B) = \cos^2(S - C).$$

37. If a spherical triangle is equal and similar to its polar triangle,

$$\sec a = \sec b \sec c + \tan b \tan c.$$

Give a particular example of such a triangle.

—(*Science and Art Exam. Papers.*)

38. Prove that by a closer approximation, employing the notation used in Legendre's Theorem (Art. 29),

$$\cos A = \cos A' - \frac{\beta\gamma\sin^2 A'}{6r^2} + \frac{\beta\gamma(\alpha^2 - 3\beta^2 - 3\gamma^2)\sin^2 A'}{180r^4},$$

39. Deduce the following formula for $a+b$ in terms of the opposite angles A and B, and the base c:—

$$\frac{a+b}{2} = \frac{c}{2} + \tan\frac{A}{2}\tan\frac{B}{2}\sin c + \frac{1}{2}\tan^2\frac{A}{2}\tan^2\frac{B}{2}\sin 2c$$
$$+ \frac{1}{3}\tan^3\frac{A}{2}\tan^3\frac{B}{2}\sin 3c + \ldots$$

—(*Bishop Law's Prize Exam., Trin. Coll., Dub.*)

[For
$$\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2},$$

(Art. 33, Ex. 9.)

being of the form $\tan\frac{a+b}{2} = n\tan\frac{c}{2}$, can be expanded in series for $\frac{a+b}{2}$.]

—(See Todhunter's *Plane Trig.*, Art. 299.)

40. From any three points on a sphere are drawn to each of three points on a great circle arcs, a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3; making angles with the great circle, respectively, A_1, A_2, A_3, etc. . . . ; prove the determinant relation

$$\begin{vmatrix} \sin a_1 \cos A_1 & \sin a_2 \cos A_2 & \sin a_3 \cos A_3 \\ \sin b_1 \cos B_1 & \sin b_2 \cos B_2 & \sin b_3 \cos B_3 \\ \sin c_1 \cos C_1 & \sin c_2 \cos C_2 & \sin c_3 \cos C_3 \end{vmatrix} = 0.$$

41. If a and b are the sides, and c the base of a triangle, prove that if θ be the angle which β, the bisector of the base, makes with the base,

$$\tan^2\tfrac{1}{2}a\tan^2\tfrac{1}{2}b = \frac{\tan^4\frac{1}{4}c - 2\tan^2\frac{1}{4}c\tan^2\frac{1}{2}\beta\cos 2\theta + \tan^4\frac{1}{2}\beta}{1 - 2\tan^2\frac{1}{4}c\tan^2\frac{1}{2}\beta\cos 2\theta + \tan^4\frac{1}{4}c\tan^4\frac{1}{2}\beta}.$$

—(*Dublin Univ. Exam. Papers.*)

[Find $\tan^2\frac{1}{2}a$, and $\tan^2\frac{1}{2}b$, and multiply the results. See Ex. 3, Paper XII].

42. Find the locus of the vertex, having given the base and—

(1) Ratio of cotangents of the base angles.

(2) Sum or difference of cotangents.

(3) $l\cot A + m\cot B$. (Cf. Art. 39, Ex. 10.)

—(*Educational Times*, April, 1885.)

CHAPTER IV.

RIGHT-ANGLED TRIANGLES.

Section I.

44. **Geometrical Deduction of Formulæ connecting the Parts of a Right-angled Triangle.**—The formulæ necessary for the solution of right-angled triangles have been deduced in Chapter III. as particular cases of the general formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

We, however, add the following geometrical method of obtaining them, as it is both simple and instructive:—

Let ABC (fig. 24) be a triangle right-angled at C, and let O be the centre of the sphere on which it is described. Since C is a right angle, the planes OCA and OCB are perpendicular to each other (Art. 6). Take R any point on the radius OC, and draw RP perpendicular to OC, and RQ perpendicular to OB. Join PQ. Since the planes COA and COB are at right angles, the angle PRQ is right, and, as in Art. 32, the angle OQP is also right. Hence we have

$$\frac{OQ}{OP} = \frac{OQ}{OR} \cdot \frac{OR}{OP};$$

that is, $$\cos c = \cos a \cos b. \tag{1}$$

$$\frac{QR}{QO} = \frac{QR}{QP} \cdot \frac{QP}{QO};$$

that is,
$$\left.\begin{aligned} \tan a &= \cos B \tan c. \\ \text{Similarly,}\quad \tan b &= \cos A \tan c. \end{aligned}\right\} \tag{2}$$

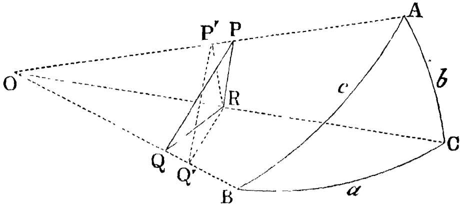

Fig. 24.

$$\frac{PR}{PO} = \frac{PR}{PQ} \cdot \frac{PQ}{PO};$$

that is,
$$\left.\begin{aligned} \sin b &= \sin B \sin c. \\ \text{Similarly,}\quad \sin a &= \sin A \sin c. \end{aligned}\right\} \tag{3}$$

$$\frac{RP}{RO} = \frac{RP}{RQ} \cdot \frac{RQ}{RO};$$

that is,
$$\left.\begin{aligned} \tan b &= \tan B \sin a. \\ \text{Similarly,}\quad \tan a &= \tan A \sin b. \end{aligned}\right\} \tag{4}$$

Again, from R draw RP' perpendicular to OA, and draw RQ' perpendicular to OC, in the plane COB, and meeting OB in Q'. Then it is clear, from the construction, that the plane of the triangle $P'Q'R$ is perpendicular

to the line OA, just as the plane of the triangle PQR is perpendicular to OB; and therefore, as in Art. 32, $OP'Q'$ is a right angle, $RP'Q' = A$; and we have

$$\cot A \cot B = \frac{RP'}{RQ'} \cdot \frac{RQ}{RP} = \frac{RP'}{RP} \cdot \frac{RQ}{RQ'}.$$

But $$\frac{RP'}{RP} = \sin RPP' = \cos ROP = \cos b,$$

and $$\frac{RQ}{RQ'} = \sin RQ'Q = \cos ROQ' = \cos a;$$

therefore $$\cot A \cot B = \cos a \cos b = \cos c. \tag{5}$$

$$\frac{\cos B}{\cos b} = \frac{QR}{QP} \div \frac{RP'}{RP} = \frac{QR}{RP'} \cdot \frac{RP}{QP} = \frac{QR}{OR} \cdot \frac{OR}{RP'} \cdot \frac{RP}{OP} \cdot \frac{OP}{QP}$$

$$= \frac{\sin a}{\sin b} \cdot \frac{\sin b}{\sin c} = \frac{\sin a}{\sin c} = \sin A;$$

$$\left.\begin{array}{ll} \text{therefore,} & \cos B = \sin A \cos b. \\ \text{Similarly,} & \cos A = \sin B \cos a. \end{array}\right\} \tag{6}$$

45. **Napier's Rules of Circular Parts.**—The formulæ of the foregoing Article enable us, being given any two parts other than the right angle, to determine all the remaining parts of the triangle. They are learned by most students separately and in general terms. Napier, however, has given two rules, called after him, which embrace them all, and which are found by some to be useful aids to the memory; their utility, however, is much questioned by others. We give them here without any investigation, considering it sufficient to merely state them.

The Student may test them by deducing all the preceding formulæ from them, and, if anxious for further information regarding them, Napier's *Mirifici Logarithmorum, Canonis Descriptio*, or Todhunter's *Spherical Trigonometry*, may be consulted.

Omitting the right angle, the two sides which include it and the complements of the remaining parts are termed the *Circular Parts* of the triangle. Thus the five circular parts are a, b, $\frac{1}{2}\pi - A$, $\frac{1}{2}\pi - c$, $\frac{1}{2}\pi - B$. The student may remark that the order in which the circular parts are here written is that in which they naturally occur in the triangle.

Divide a circle (fig. 25) into five sectors, and on each sector write one of the circular parts, taking care to write them round the circle in the order they are written above; that is, in the order in which they occur in the triangle.

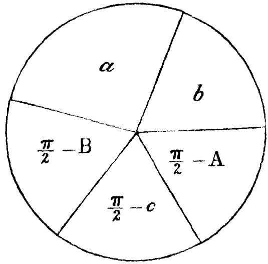

Fig. 25.

Any one of the parts being selected and called the *middle part*, the two parts contiguous to it are called the *adjacent parts*, and the remaining two are called the *opposite parts.* Thus, for example, if $\frac{1}{2}\pi - c$ be selected

for middle part, then $\frac{1}{2}\pi - A$ and $\frac{1}{2}\pi - B$ will be the adjacent parts, and a and b will be the opposite parts.

This being premised, Napier's Rules are the following:—

I. Sine (middle part) = product of tangents of adjacent parts.

II. Sine (middle part) = product of cosines of opposite parts.

45 (a). **Further Fundamental Relations.**—In addition to the foregoing formulæ the following results, analogous to those of Euc. VI., VIII., are of importance in the theory of right-angled triangles. Let a perpendicular be drawn to the hypotenuse of a right-angled triangle, from the opposite vertex, then the segments of the hypotenuse are connected with the sides, and the perpendicular by the relations.

(1) *The tangent of either side is a geometric mean between the tangent of the adjacent segment and the tangent of the hypotenuse.*

(2) *The sine of the perpendicular is a geometric mean between the tangents of the segments of the hypotenuse.*

For if p be the perpendicular, α and β the segments of the hypotenuse adjacent to the sides a and b, we have, by Art. 44 (2),

$$\cos B = \frac{\tan a}{\tan c} = \frac{\tan \alpha}{\tan a}.$$

Therefore, $\tan^2 a = \tan \alpha \tan c.$

Similarly, $\tan^2 b = \tan \beta \tan c.$

Again, by Art. 44 (4),

$$\tan \alpha = \sin p \tan \widehat{pa},$$

and $$\tan \beta = \sin p \tan \widehat{pb}.$$

But $\widehat{pa}$, the angle between p and a, is the complement of $\widehat{pb}$. Therefore, by multiplication, we obtain

$$\tan \alpha \tan \beta = \sin^2 p.$$

45 (b). **Quadrantal Triangles.**—When one side of a triangle is a quadrant it is termed a *Quadrantal Triangle.* The polar reciprocal of such a triangle is obviously right-angled, and the solution of the quadrantal is thus reduced at once to that of the right-angled triangle. Hence, the parts of a quadrantal triangle are connected by the following formulæ, supplemental to those of Art. 44:—

(1) $\cos C = - \cos A \cos B,$

(2) $\tan A = - \cos b \tan C,$

(3) $\sin A = \sin a \sin C,$

(4) $\tan A = \tan a \sin B,$

(5) $\cos C = - \cot a \cot b,$

(6) $\cos a = \cos A \sin b.$

Examples.

1. Show that formula (1), Art. 44, reduces to Euc. I. XLVII. when the radius of the sphere is infinitely great.

[This appears at once on substituting the formulæ of Art. 18, and reducing.]

2. Show that formula (5) reduces to an identity *in plano*. Similarly, also formula (6) reduces to an identity.

3. Formulæ (2), (3), and (4) reduce to the ordinary expressions for the cosine, sine, and tangent of a plane angle respectively.

4. If c be less than 90°, show that a and b are of like affection.*

[In this case $\cos c$ is positive; and hence $\cos a$ and $\cos b$ must have like signs, as appears from formula (1).]

5. If c be greater than 90°, a and b are of opposite affection.

6. The sides of a right-angled triangle must be each less than a quadrant, or two of them must be each greater than a quadrant.

[This follows on combining Examples 4 and 5.]

7. Show that a side and the hypotenuse of a right-angled triangle are of the same or opposite affection, according as the included angle is less than, or greater than, a right angle.

[Apply formula (2), Art. 44.]

8. The sides of a right-angled triangle are of the same affection as the opposite angles.

[By formula (6), $\cos a = \dfrac{\cos A}{\sin B}$; and therefore $\cos a$ is positive or negative, according as $\cos A$ is positive or negative.]

9. If one of the sides of a right-angled triangle be equal to the opposite angle, the remaining parts are each equal to 90°.

[See Art. 36, Ex. 9.]

10. If the angle A of a right-angled triangle be acute, show that the difference of the sides which contain it is less than 90°.

[The opposite side (a) is less than 90°, but it is greater than the difference between the other two; therefore $c - b < 90°$.]

---

* Two angles are said to be of the same affection when they are each less than, or each greater than, a right angle.

Section II.

Solution of Right-angled Triangles—Numerical Examples.

46. **Solution of Triangles.**—It has been already stated that every spherical triangle has six *parts*—namely, the three angles and the three sides, and that when any three of the parts are given, the remaining three can be determined.

The process by which the remaining three are determined is called the *solution* of the triangle.

In the practical solution of triangles the numerical calculations are performed by aid of a table of logarithms. Hence, if we have at our disposal two or more formulæ, from which any required part can be determined in terms of other known parts, we should use that formula which is most readily adapted to logarithmic computation. If it should happen that a required part can only be obtained from a formula not adapted to logarithmic computation, it can be found by throwing the formula into logarithmic form by the use of a subsidiary angle. Any notice of this process is postponed, however, for the present. (See Art. 76.)

We now proceed to the solution of all the cases which can be presented. They are six in number, and, as before remarked, we require to know only two parts, the right angle forming a constant known part.

Remark.—In determining any of the parts, the use of the parts previously determined should be carefully

avoided, lest they should be erroneous. Each part should be determined from the given parts alone.

It is always well to test the accuracy of the results by using the parts obtained to determine one of the given parts.

47. Case I.—*Having given the two sides a and b*, to determine A, B, and c, we have—

For A,	$\cot A = \cot a \sin b$.	[Formula (4), Art. 44.]
For B,	$\cot B = \cot b \sin a$.	,, ,, ,,
For c,	$\cos c = \cos a \cos b$.	,, (1), ,,

In this case the solution is unique, as there is manifestly no ambiguity, none of the parts being determined from their sines.

Examples.

1. Given $a = 54° \, 16'$ and $b = 33° \, 12'$.

For A, $10 + L \cot A = L \cot (54° \, 16') + L \sin (33° \, 12')$;

therefore $L \cot A = 9 \cdot 5954382$,

and $A = 68° \, 29' \, 53''$.

For B, $10 + L \cot B = L \cot (33° \, 12') + L \sin (54° \, 16')$;

therefore $L \cot B = 10 \cdot 0935879$,

and $B = 38° \, 52' \, 26''$.

For c, $10 + L \cos c = L \cos a + L \cos b$;

therefore $L \cos c = 9 \cdot 6890261$,

and $c = 60° \, 44' \, 46''$.

2. Given $a = 55° \; 18'$ and $b = 39° \; 27'$.

$$L \cot A = 9{\cdot}6434280 \; ; \; \therefore \; A = 66° \; 15' \; 6''.$$
$$L \cot B = 9{\cdot}9996157 \; ; \; \therefore \; B = 45° \; 1' \; 31''.$$
$$L \cos c = 9{\cdot}6430438 \; ; \; \therefore \; c = 63° \; 55' \; 21''.$$

3. Given $a = 56° \; 20'$ and $b = 78° \; 40'$.

$$L \cot A = 9{\cdot}8149722 \; ; \; \therefore \; A = 56° \; 51' \; 7''.$$
$$L \cot B = 9{\cdot}2222192 \; ; \; \therefore \; B = 80° \; 31' \; 48''.$$
$$L \cos c = 9{\cdot}0371914 \; ; \; \therefore \; c = 83° \; 44' \; 44\tfrac{1}{2}''.$$

48. Case II.—*Having given the two angles A and B,* to determine a, b, and c, we have—

For a, $\cos A = \cos a \sin B$. [Formula (6), Art. 44.]

For b, $\cos B = \cos b \sin A$. ,, ,, ,,

For c, $\cos c = \cot A \cot B$. ,, (5), ,,

In this case, as in Case I., there is no ambiguity.

Examples.

1. Given $A = 74° \; 15'$ and $B = 32° \; 10'$.

For a,
$$L \cos a = 10 + L \cos (74° \; 15') - L \sin (32° \; 10') = 9{\cdot}7074497 \; ;$$
therefore $a = 59° \; 20' \; 44''$.

For b,
$$L \cos b = 10 + L \cos (32° \; 10') - L \sin (74° \; 15') = 9{\cdot}9442480 \; ;$$
therefore $b = 28° \; 24' \; 54''$.

For c,
$$10 + L \cos c = L \cot A + L \cot B.$$
$$L \cos c = 9{\cdot}6516976 \; ;$$
therefore $c = 63° \; 21' \; 24\tfrac{1}{2}''$.

2. Given $A = 52° 26'$ and $B = 49° 15'$.

$$L \cos a = 9{\cdot}9056850; \therefore a = 36° 24' 34{\cdot}5''.$$

$$L \cos b = 9{\cdot}9156750; \therefore b = 34° 33' 40''.$$

$$L \cos c = 9{\cdot}8213599; \therefore c = 48° 29' 20''.$$

3. Given $A = 64° 15'$ and $B = 48° 24'$.

$$L \cos a = 9{\cdot}7641507; \therefore a = 54° 28' 53''.$$

$$L \cos b = 9{\cdot}8675405; \therefore b = 42° 30' 47''.$$

$$L \cos c = 9{\cdot}6316912; \therefore c = 64° 38' 38''.$$

49. Case III.—*Having given the hypotenuse c and a side a*, to determine A, B, and b, we have—

For A, $\sin A = \sin a \div \sin c$. [Formula (3), Art. 44.]

For B, $\cos B = \tan a \div \tan c$. ,, (2), ,,

For b, $\cos b = \cos c \div \cos a$. ,, (1), ,,

In this case there is an apparent ambiguity in the value of A, but this is removed by the consideration that A and a are always of the same affection. (See Ex. 8, Art. 45).

Examples.

1. Given $c = 72° 30'$ and $a = 45° 15'$.

For A, $L \sin A = 10 + L \sin (45° 15') - L \sin (72° 30')$
$= 9{\cdot}8719522$;

therefore $A = 48° 7' 44{\cdot}5''$.

For B, $L \cos B = L \tan (45° 15') - L \tan (72° 30') + 10$
$= 9{\cdot}5025123$;

therefore $B = 71° 27' 15''$.

For b, $L \cos b = L \cos (72° 30') - L \cos (45° 15') + 10$
$= 9{\cdot}6305601$;

therefore $b = 64° 42' 52''$.

2. Given $c = 76° 42'$ and $a = 47° 18'$.

$L \sin A = 9 \cdot 8780442$; $\therefore A = 49° \ 2' \ 24 \cdot 5''$.

$L \cos B = 9 \cdot 4085340$; $\therefore B = 75° \ 9' \ 24 \cdot 75''$.

$L \cos b = 9 \cdot 5304897$; $\therefore b = 70° \ 10' \ 13''$.

3. Given $c = 91° 18'$ and $a = 72° 27'$.

$L \sin A = 9 \cdot 9794116$; $\therefore A = 72° \ 29' \ 48''$.

$L \cos$ sup. of $B = 8 \cdot 8558531$; $\therefore B = 94° \ 6' \ 53 \cdot 3''$.

$L \cos$ sup. of $b = 8 \cdot 8764415$; $\therefore b = 94° \ 18' \ 53 \cdot 8''$.

50. CASE IV.—*Having given the hypotenuse c and an angle A*, to determine a, b, and B, we have—

For a,	$\sin a = \sin A \sin c$.	[Formula	(3),	Art. 44.]
For b,	$\tan b = \cos A \tan c$.	"	(2),	"
For B,	$\cot B = \tan A \cos c$.	"	(5),	"

The apparent ambiguity in the value of a is removed, as in CASE III., by the consideration that a and A are of the same affection.

Examples.

1. Given $c = 84° 20'$ and $A = 35° 25'$.

For a, $L \sin a = L \sin (35° 25') + L \sin (84° 20') - 10$

$= 9 \cdot 7609396$;

therefore $a = 35° \ 13' \ 4''$.

For b, $L \tan b = L \cos (35° 25') + L \tan (84° 20') - 10$

$= 10 \cdot 9145116$;

therefore $b = 83° \ 3' \ 29''$.

For B, $L \cot B = L \tan (35° 25') + L \cos (84° 20') - 10$

$= 8 \cdot 8464279$;

therefore $B = 85° \ 59' \ 1''$.

2. Given $c = 67° 54'$ and $A = 43° 28'$.

$L \sin a = 9 \cdot 8044046$; $\therefore a = 39° 35' 51''$.

$L \tan b = 10 \cdot 2522138$; $\therefore b = 60° 46' 25\frac{1}{2}''$.

$L \cot B = 9 \cdot 5521908$; $\therefore B = 70° 22\ 21''$.

3. Given $c = 22° 18' 30''$ and $A = 47° 39' 36''$.

$L \sin a = 9 \cdot 4480545$; $\therefore a = 16° 17' 41''$.

$L \tan b = 9 \cdot 4414574$; $\therefore b = 15° 26' 53''$.

$L \cot B = 10 \cdot 0065973$; $\therefore B = 44° 33' 53 \cdot 4''$.

51. Case V.—*Having given a side a and the adjacent angle B*, to determine A, b, and c, we have—

For A, $\cos A = \cos a \sin B$. [Formula (6), Art. 44.]

For b, $\tan b = \tan B \sin a$. „ (4), „

For c, $\cot c = \cot a \cos B$. „ (2), „

In this case there is evidently no ambiguity.

Examples.

1. Given $a = 31° 20' 45''$ and $B = 55° 30' 30''$.

For A, $L \cos A = L \cos (31° 20' 45'') + L \sin (55° 30' 30'') - 10$
$= 9 \cdot 8475168$;

therefore $A = 45° 15' 30 \cdot 6''$.

For b, $L \tan b = L \tan (55° 30' 30'' + L \sin (31° 20' 45'') - 10$
$= 9 \cdot 8791734$;

therefore $b = 37° 7' 50''$.

For c, $L \cot c = L \cot (31° 20' 45'') + L \cos (55° 30' 30'') - 10$
$= 9 \cdot 9683434$;

therefore $c = 47° 5' 11''$.

2. Given $a = 82° 6'$ and $B = 43° 28'$.

$$L \cos A = 8{\cdot}9756733; \quad \therefore A = 84° \; 34' \; 28''.$$
$$L \tan b = 9{\cdot}9726026; \quad \therefore b = 43° \; 11' \; 38''.$$
$$L \cot c = 9{\cdot}0030707; \quad \therefore c = 84° \; 14' \; 57''.$$

3. Given $a = 42° \; 30' \; 30''$ and $B = 53° \; 10' \; 30''$.

$$L \cos A = 9{\cdot}7709180; \quad \therefore A = 53° \; 50' \; 12''.$$
$$L \tan b = 9{\cdot}9554001; \quad \therefore b = 42° \; 3' \; 47''.$$
$$L \cot c = 9{\cdot}8155179; \quad \therefore c = 56° \; 49' \; 8''.$$

52. Case VI.—*Having given a side a and the opposite angle A*, to determine b, c, and B, we have—

For b,	$\sin b = \tan a \cot A$.	[Formula (4), Art. 44.]
For c,	$\sin c = \sin a \div \sin A$.	,, (3), ,,
For B,	$\sin B = \cos A \div \cos a$.	,, (6), ,,

In this case an ambiguity arises as the parts are determined from their sines, and two values for each are in general admissible. However, for each value of b there will, in general, be only one value for c, since c and b are connected by the relation $\cos c = \cos a \cos b$; and at the same time only one admissible value for B, since $\cos c = \cot A \cot B$. Hence there will in general be only *two* triangles having the given parts, except when the side a is a quadrant and the angle $A = 90°$, in which case the solution becomes *indeterminate*.

When $a = A$, the formulæ, and also the figure, show that b, c, and B are right angles, and the solution is therefore *unique*.

It is easily seen that a is less than A when both are acute (they must be of the same affection), and greater than A when both are obtuse; and when the data do not satisfy these conditions the solution is impossible.

It appears at once from the figure that there are in general two solutions, for the *colunar* triangle $A'BC$ has also the given parts A' and a, and therefore satisfies the given conditions.

Examples.

1. Given $a = 46° \; 45'$ and $A = 59° \; 12'$.

For b, $L \sin b = L \tan (46° \; 45') + L \cot (59° \; 12') - 10$
$= 9 \cdot 8018795$;

therefore $b = 39° \; 19' \; 23\tfrac{1}{2}''$, or $140° \; 40' \; 36\tfrac{1}{2}''$.

For c, $L \sin c = L \sin a - L \sin A + 10$
$= 9 \cdot 9283797$;

therefore $c = 57° \; 59' \; 29''$, or $122° \; 0' \; 31''$.

For B, $L \sin B = L \cos A - L \cos a + 10$
$= 9 \cdot 8734997$;

therefore $B = 48° \; 21' \; 28''$, or $131° \; 38' \; 32''$.

2. Given $a = 21° \; 39'$ and $A = 42° \; 10' \; 10''$.

$L \sin b = 9 \cdot 6417030$; $\therefore b = 25° \; 59' \; 27 \cdot 8''$, or $154° \; 0' \; 32 \cdot 2''$.

$L \sin c = 9 \cdot 7400177$; $\therefore c = 33° \; 20' \; 13 \cdot 4''$, or $146° \; 39' \; 46 \cdot 6''$.

$L \sin B = 9 \cdot 9016853$; $\therefore B = 52° \; 23' \; 2 \cdot 8''$, or $127° \; 36' \; 57 \cdot 2''$.

3. Given $a = 25° \; 18' \; 45''$ and $A = 15° \; 58' \; 15''$.

[*Ans.* Impossible by Art. 52.]

Section III.

Theorems.

53. In this section we propose to investigate some theorems which apply to spherical triangles in general, but which are immediately deducible from a knowledge of the properties of right-angled triangles.

54. **Theorem.**—*The arcs drawn from the vertices of a spherical triangle perpendicular to the opposite sides are concurrent.*

Let AP, BQ (fig. 26), be arcs drawn from A, B, per-

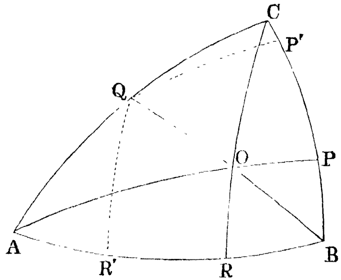

Fig. 26.

pendicular to BC and CA, respectively, and let O be their point of intersection.

Now, $$\frac{\sin OQ}{\sin OR} = \frac{\sin OAQ}{\sin OAR} = \frac{\sec B}{\sec C};$$

[Formula (6), Art. 44.]

and it is obvious that the sines of the perpendiculars from any other point of AP on the sides b and c bear the same ratio to each other.

Similarly, $$\frac{\sin OP}{\sin OR} = \frac{\sec A}{\sec C};$$

and hence, on dividing this equation by the former, we get

$$\frac{\sin OP}{\sin OQ} = \frac{\sec A}{\sec B},$$

and therefore the arc joining C to O is perpendicular to the side AB. Thus the perpendiculars meet in a point; and if α, β, γ be the lengths intercepted on them between their point of concurrence and the sides a, b, and c, respectively, we have

$$\sin \alpha : \sin \beta : \sin \gamma = \sec A : \sec B : \sec C.$$

In the same manner, it may be shown that the arcs joining the vertices to the middle points of the opposite sides are concurrent; and if α', β', γ' be the perpendiculars from their point of concurrence on the sides a, b, c, we have

$$\sin \alpha' : \sin \beta' : \sin \gamma' = \operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C.$$

Similarly for the point of intersection of perpendiculars at middle points of sides—

$$\sin \alpha'' : \sin \beta'' : \sin \gamma'' = \sin (S - A) : \sin (S - B) : \sin (S - C.)$$

55. *If from three points A, B, C on a great circle perpendiculars AA', BB', CC' be drawn to another great circle—*

(1) $\sin BC \sin AA' + \sin CA \sin BB' + \sin AB \sin CC' = 0,$
(2) $\sin B'C' \tan AA' + \sin C'A' \tan BB' + \sin A'B' \tan CC' = 0,$
(3) $\sin BC \cot A + \sin CA \cot B + \sin AB \cot C = 0,$
(4) $\sin B'C' \cos A + \sin C'A' \cos B + \sin A'B' \cos C = 0,$

where A, B, C are the angles the perpendiculars AA', BB', CC' make with the great circle ABC.

To prove (1)—

$$\frac{\sin AA'}{\sin AD} = \frac{\sin BB'}{\sin BD} = \frac{\sin CC'}{\sin CD} = \sin D \text{ (fig. 27)}. \qquad (\alpha)$$

But since A, B, C are concyclic (Art. 12, Ex. 15),

$$\sin BC \sin AD + \sin CA \sin BD + \sin AB \sin CD = 0$$

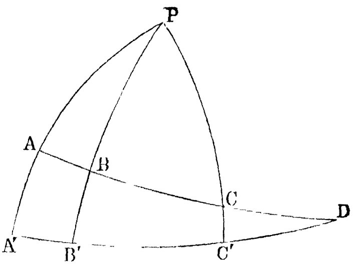

Fig. 27.

Substituting in this expression the values of sin AD, sin BD, and sin CD, obtained from the equations (α), we obtain (1). [Cf. Art. 40.]

To prove (2)—

$$\frac{\tan AA'}{\sin A'D} = \frac{\tan BB'}{\sin B'D} = \frac{\tan CC'}{\sin C'D} = \tan D. \qquad (\beta)$$

But, as before,

$$\sin B'C' \sin A'D + \sin C'A' \sin B'D + \sin A'B' \sin C'D = 0.$$

Substituting in this expression the values of sin $A'D$, sin $B'D$, and sin $C'D$, obtained from the equations (β), we obtain (2).

To prove (3)—

$$\frac{\cos AD}{\cot A} = \frac{\cos BD}{\cot B} = \frac{\cos CD}{\cot C} = \cot D.$$

But

$$\sin BC \cos AD + \sin CA \cos BD + \sin AB \cos CD = 0;$$

therefore, &c. [See Chap. III., Misc. Ex. 42.]

To prove (4)—

$$\frac{\cos A}{\cos A'D} = \frac{\cos B}{\cos B'D} = \frac{\cos C}{\cos C'D} = \sin D.$$

[Formula (6), Art. 44.]

But

$$\sin B'C' \cos A'D + \sin C'A' \cos B'D + \sin A'B' \cos C'D = 0;$$

therefore, &c.

By observing that AA', BB', and CC' meet at P, the pole of $A'B'C'$, we are enabled to write down the following corollaries:—

Cor. 1.—If B be a point on the base AC of a triangle APC, we have

$$\sin BC \cos AP + \sin CA \cos BP + \sin AB \cos CP = 0.$$

[Cf. Art. 40.]

Cor. 2.—

$$\cot AP \sin BPC + \cot BP \sin CPA + \cot CP \sin APB = 0.$$

[Cf. Art. 39, Ex. 10.]

Cor. 3.—

$$\cos A \sin BPC + \cos B \sin CPA + \cos C \sin APB = 0.$$

[Cf. Art. 35, Ex. 5.]

NOTE.—It may be noticed that (1) and (4) are supplemental theorems, as are also (2) and (3). When an arc is drawn from the vertical angle of a triangle to meet the base, (1) connects it with the segments of the base; (2) connects it with the segments of the vertical angle; (3) connects the angle it makes with the base with the segments of the base; and (4) connects the angle it makes with the base with the segments of the vertical angle, the parts of the triangle being supposed known.

56. **Geometrical Deduction of the Analogies of Napier and Delambre.**—*Construction.* On the base AB of the triangle ABC (fig. 28) construct an isosceles triangle AOB, having each base angle $= \frac{1}{2}(A + B)$. From its vertex O draw OX, OY, OZ perpendiculars to the sides a, b, and c of the given triangle. Join OC.

Now the triangles BOX and AOY are equal in all respects; for the side BO and angle X are equal to the side AO and the angle Y, respectively; and the angle $OBX = B - \frac{1}{2}(A + B) = \frac{1}{2}(B - A) =$ angle OAY. Therefore the triangles are equal in all respects, and as a consequence the triangles COX and COY are equal in all respects.

Hence, $$AY = BX = \tfrac{1}{2}(a + b),$$

and $$CX = CY = \tfrac{1}{2}(a - b).$$

But since the angles AOY and BOX are equal, it follows

that the angles AOB and XOY are equal, and hence their halves AOZ and COY are equal.

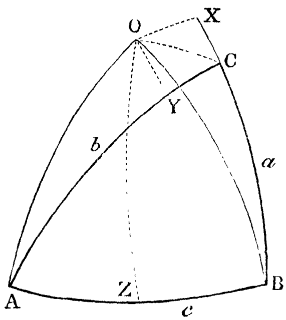

Fig. 28.

Firstly—*Napier's Analogies**:

1. Comparing the triangles AOY and COY, we have

$$\frac{\cos AO}{\cos CO} = \frac{\cos AY}{\cos CY}; \qquad \text{[Art. 44, (1).]}$$

and comparing the triangles, AOZ and COY, we have, also,

$$\frac{\cos AO}{\cos CO} = \frac{\cot OAZ}{\cot OCY}. \qquad \left[\begin{array}{c}\text{Art. 44 (5)}\\ \text{Since } AOZ = COY.\end{array}\right]$$

Therefore,

$$\frac{\tan OAZ}{\tan OCY} = \frac{\cos CY}{\cos AY},$$

* This method of obtaining Napier's Analogies is suggested by, and strictly analogous to, the geometrical method of arriving at the formula $\frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{a-b}{a+b}$ in *Plane Trigonometry*. The same principle is used in the *Encyc. Brit.*, Article, "Trigonometry."

or $$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C.$$

2. $\tan OY = \tan OAY \sin AY = \tan OCY \sin CY$. [Art. 44 (4).]

Hence,

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C.$$

3. $$\tan AO = \frac{\tan AZ}{\cos OAZ} = \frac{\tan AY}{\cos OAY}.$$ [Art. 44 (2).]

Hence

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c.$$

4. Since $COY = AOZ$, we have [Art. 44 (4).]

$$\frac{\tan CY}{\sin OY} = \frac{\tan AZ}{\sin OZ}.$$

Therefore,

$$\tan CY = \frac{\sin OY}{\sin OZ} \tan AZ = \frac{\sin OAY}{\sin OAZ} \tan AZ,$$

or

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c.$$

Secondly—*Delambre's* Analogies.*

1. $$\frac{\cos OAZ}{\cos OCY} = \frac{\cos OZ}{\cos OY}$$ [Art. 44 (6).]

$$= \frac{\cos AY}{\cos AZ}.$$ [Art. 44 (1).]

---

* Delambre's Analogies are commonly called Gauss's Formulæ. (See foot note, Art. 33.)

These results may likewise be arrived at by taking O on the other side of the base as the vertex of an isosceles triangle AOB, whose base angles are each $\frac{1}{2}\pi - \frac{1}{2}(A+B)$. Cf. Art. 19 (12).

Therefore

$$\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.$$

2. $$\frac{\cos OAY}{\cos OCY}=\frac{\sin AOY}{\sin COY}\qquad\text{[Art. 44 (6).]}$$

$$=\frac{\sin AOY}{\sin AOZ}=\frac{\sin AY}{\sin AZ}.\qquad\text{[Art. 44 (3).]}$$

Therefore

$$\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.$$

3. $$\cos AOZ=\cos AZ\sin OAZ=\cos CY\sin OCY.$$

[Art. 44 (6).]

Hence

$$\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C.$$

4. $$\frac{\sin OAY}{\sin OCY}=\frac{\sin OC}{\sin OA}=\frac{\sin CY}{\sin AZ}.$$

[Art. 44 (3).]

Hence

$$\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C.$$

Examples on Fundamental Formulæ (1) to (6).

1. Prove that, if $C=90^\circ$,

$$\tan\frac{B}{2}=\frac{\sin(s-a)}{\sin s}.\qquad\text{[Cf. Art. 33, Ex. 4.]}$$

2. Prove that—

(*a*) $2n=\sin a\sin b$. [Art. 30 (3).]

(*b*) $2N=\sin a\sin B$. [Art. 36 (12).]

(*c*) $\dfrac{n}{N}=\sin c$. [Art. 36, Ex. 2.]

(*d*) $$\frac{n^2}{N^2}=\frac{\sin^2 a\sin^2 b\sin^2 c}{\sin^2 a+\sin^2 b-\sin^2 c}=\frac{\sin^2 a+\sin^2 b-\sin^2 c}{\sin^2 A-\cos^2 B}.$$

[Cf. Art. 36, Ex. 7.]

3. Prove that

$$\sin^2 a \sin^2 b = \sin^2 a + \sin^2 b - \sin^2 c. \qquad \text{[Art. 44 (1).]}$$

4. Prove that

$$\tan^2 \frac{A}{2} = \frac{\sin(c-b)}{\sin(c+b)}.$$

[This is obviously another form of (2), Art. 44.]

5. What is the Great Circle course and distance between two places in latitude 45° North, with 90° difference of longitude?

Ans. Course makes $\cos^{-1}\dfrac{1}{\sqrt{3}}$ with meridian. Distance = 60° or $4166\frac{2}{3}$ miles.

6. If p be the perpendicular from the right angle on the hypotenuse, prove that—

(*a*) $\cos^2 p = \cos^2 A + \cos^2 B.$ [Art. 44 (6).]

(*b*) $\cot^2 p = \cot^2 a + \cot^2 b.$ [Art. 44 (4).]

(*c*) $\sin^2 p \sin^2 c = \sin^2 a + \sin^2 b - \sin^2 c.$ [See Ex. 3.]

7. Express the angle A in terms of—(1) b and c; (2) c and a; (3) a and b.

8. Show that

$$2\sin^2 \tfrac{1}{2} c = \sin^2 \tfrac{1}{2}(a+b) + \sin^2 \tfrac{1}{2}(a-b).$$

[Reduces to (1), Art. 44.]

9. In a spherical triangle, if $c = 90°$, prove that

$$\tan a \tan b + \sec C = 0.$$

[Supplemental to (5), Art. 44.]

10. If the side c of a spherical triangle be a quadrant, show that

$$\sin^2 p = \cot\theta \cot\phi,$$

where p is the perpendicular on c, and θ and ϕ are the segments of the vertical angle.

11. Show that the ratio of the cosines of the segments of the base made by the perpendicular from the vertex is equal to the ratio of the cosines of the sides.

[Apply (1), Art. 44.]

12. Given the base and the ratio of the cosines of the sides of any spherical triangle; find the locus of the vertex, and state the analogue *in plano*.

[See Ex. 11.]

13. A station is in latitude λ (North), and longitude l (East); find the longitudes of the places on the Equator distant δ from the station.

—[*Science and Art Exam. Papers.*]

Ans. Longitudes $= l \pm \cos^{-1}\left(\dfrac{\cos\delta}{\cos\lambda}\right)$.

14. Given the base and base angles of any spherical triangle; calculate the segments x and $c - x$ of the base made by the perpendicular on it from the vertex. *Ans.* $\tan A \sin x = \tan B \sin (c - x)$.

15. What is the shortest distance between two places on the Earth's surface—one in latitude 45° North and longitude 45° West; the other in latitude 45° South and longitude 45° East?

Ans. 120° or 8333·3 miles.

16. Given in a right-angled triangle the vertex fixed, and

$$\cot^2 a + \cot^2 b = \text{const.};$$

find the envelope of the base. [See Ex. 6 (*b*).]

17. Show that the ratio of the cosines of the base angles is equal to the ratio of the sines of the segments of the vertical angle made by the perpendicular drawn from it to the opposite side.

18. Given the vertical angle and ratio of cosines of the base angles; find the envelope of the base.

Ans. A fixed point 90° from the vertex; its position being obtained from Ex. 17.

19. If $A = 36°$ and $B = 60°$, show that—

(1). $a + b + c = 90°$.

(2). The perpendicular on the hypotenuse $= 18°$.

(3). $\cot a - \cot b = 1$.

20. If λ and δ be perpendiculars from a point S on two great circles intersecting at an angle ω, and if the intercepts they make on them from their point of intersection be l and a, show that

$$\cos\omega = \frac{\sin\lambda \tan l + \sin\delta \tan a}{\sin\lambda \tan a + \sin\delta \tan l}.$$

—[Professor Haughton, *Educational Times.*]

[Apply the formulæ of Art. 37 to the triangle SPK, where P and K are the poles of the great circles, and eliminate $\sin\omega$.]

21. Using the notation of Ex. 20, show that—

(1). $$\frac{\cos\frac{1}{2}(\omega-\omega')}{\cos\frac{1}{2}(\omega+\omega')}=\frac{\tan\frac{1}{2}\lambda\tan\frac{1}{2}l+\tan\frac{1}{2}\delta\tan\frac{1}{2}a}{\tan\frac{1}{2}l\tan\frac{1}{2}\delta+\tan\frac{1}{2}\lambda\tan\frac{1}{2}a}.$$

(2). $$\frac{\sin\frac{1}{2}(\omega-\omega')}{\sin\frac{1}{2}(\omega+\omega')}=\frac{\tan\frac{1}{2}\delta\tan\frac{1}{2}\lambda-\tan\frac{1}{2}l\tan\frac{1}{2}a}{1-\tan\frac{1}{2}a\tan\frac{1}{2}l\tan\frac{1}{2}\delta\tan\frac{1}{2}\lambda},$$

ω' being the angle between δ and λ.

Miscellaneous Examples.

In the following Examples the triangle is supposed right-angled (at C), unless otherwise stated:—

1. If β be the bisector of the hypotenuse of a right-angled triangle, show that

$$\sin^2\beta=\frac{\sin^2 a+\sin^2 b}{4\cos^2\frac{c}{2}}.$$

[By Art. 42—

$$\cos a+\cos b=2\cos\beta\cos\frac{c}{2}.$$

Squaring, we get—

$$\cos^2 a+\cos^2 b+2\cos c=2(1-\sin^2\beta)(1+\cos c);\ \therefore\ \&c.]$$

2. If ι and η be the internal and external bisectors of C, prove that—

$$(a)\ 2\cot^2\iota=(\cot a+\cot b)^2,$$

$$(b)\ 2\cot^2\eta=(\cot a-\cot b)^2.$$ [See Art. 39, Ex. 1.]

3. Show that—

$$(a)\ \cot^2\iota+\cot^2\eta=\cot^2 a+\cot^2 b,$$

$$(b)\ \cot^2\iota-\cot^2\eta=2\cot a\cot b.$$

4. If the bisectors of the vertical angle C meet the base in X and Y, show that

$$2\cos XY=(\cot^2 a-\cot^2 b)\sin\iota\sin\eta.$$

[Cf. Art. 39, Ex. 6.]

5. If ϕ be the angle between the bisector of the vertical angle and the perpendicular, show that

$$\tan\phi=\frac{\sin(a-b)}{\sin(a+b)}.$$

[See Chap. III., Misc. Ex. 11.]

6. In a right-angled triangle, show that

$$\sin(A+B) = \frac{\cos a + \cos b}{1 + \cos a \cos b}.$$

[Apply Delambre's Formulæ.]

7. Show that

$$\tan S = \cot\frac{a}{2}\cot\frac{b}{2}.$$

Here
$$\tan S = \frac{1 + \tan\dfrac{A+B}{2}}{1 - \tan\dfrac{A+B}{2}}.$$

[Apply Napier's Analogies.]

8. Construct a right-angled triangle, being given the hypotenuse and—

(1) The sum of the base angles;
(2) The difference of the base angles.

[Apply Gauss's Formulæ.]

9. Given the hypotenuse and the sum or difference of the sides; construct the triangle. (See Ex. 8.)

10. Given the sum of the sides a and b, and the sum of the base angles; solve the triangle.

[Apply Gauss's Formulæ.]

11. Show that

$$\sin\frac{A}{2} = \frac{\sqrt{\sin c + \sin a} + \sqrt{\sin c - \sin a}}{2\sqrt{\sin c}}.$$

[Here, if we divide out by $\sqrt{\sin c}$, and put $\dfrac{\sin a}{\sin c} = \sin A$, the right-hand side becomes $\frac{1}{2}(\sqrt{1 + \sin A} + \sqrt{1 - \sin A})$. But $1 \pm \sin A = \left(\sin\dfrac{A}{2} \pm \cos\dfrac{A}{2}\right)^2$; therefore, &c.]

12. If α, α', β, β', γ, γ', be the segments of the sides of any spherical triangle made by the perpendiculars from the opposite vertex, show that

$$\cos\alpha\cos\beta\cos\gamma = \cos\alpha'\cos\beta'\cos\gamma'.$$

[For
$$\frac{\cos\alpha}{\cos\alpha'} = \frac{\cos a}{\cos b};\quad \frac{\cos\beta}{\cos\beta'} = \frac{\cos b}{\cos c};\quad \frac{\cos\gamma}{\cos\gamma'} = \frac{\cos c}{\cos a}.$$

(See Art. 56, Ex. 11). Hence $\dfrac{\cos\alpha\cos\beta\cos\gamma}{\cos\alpha'\cos\beta'\cos\gamma'} = 1$; therefore, &c.]

13. Using the notation of the preceding Example, show that—

$$(a)\ \sin\alpha \sin\beta \sin\gamma = \sin\alpha' \sin\beta' \sin\gamma',$$

$$(b)\ \tan\alpha \tan\beta \tan\gamma = \tan\alpha' \tan\beta' \tan\gamma'.$$

[(*a*) follows from the formula $\tan p = \tan A \sin a = \tan B \sin a'$.]

14. Find the distance between any two points on the Earth's surface, given their latitudes and longitudes. (*London Univ. Exam. Papers.*)

If λ and λ' be their latitudes, and l and l' their longitudes, we have

$$\cos\delta = \sin\lambda \sin\lambda' + \cos\lambda \cos\lambda' \cos(l - l').$$

15. The cosines of the base angles of any triangle are proportional to the cosines of angles which the sides make with the arc joining the vertex to the pole of the perpendicular from the vertex on the base.

[For these latter cosines are equal to the sines of the segments of the vertical angle made by the perpendicular.]

16. If δ be the length of the arc through the vertex of an isosceles triangle, dividing the base into segments α and β, prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \tan\frac{a+\delta}{2}\tan\frac{a-\delta}{2},$$

where a is one of the equal sides of the triangle.

[See Art. 43, or apply Napier's Analogies.]

17. Find a relation connecting the mutual distances of any three places with their latitudes.

[By the aid of Ex. 14. See also Art. 55 (1).]

18. On the Earth considered as a sphere a great circle is drawn, inclined to the Equator at an angle of 30°, and cutting it in points whose longitudes are 0° and 180°. Find the longitudes of the meridians, dividing the great circle into eight equal parts.

19. Two points A and B are taken on two secondaries to a great circle $A'B'$. If θ be the angle between the arcs AB and $A'B'$, its value is found by the equation

$$\sin A'B' = \frac{\sin AB \cos\theta}{\cos AA' \cos BB'}. \quad (Q.\ U.\ I.\ Exam.\ Papers.)$$

$$\left[\frac{\cos\theta}{\cos AA' \cos BB'} = \frac{\sin A}{\cos BB'} = \frac{\sin A}{\sin BC} = \frac{\sin A'B'}{\sin AB};\ \text{therefore, \&c.}\right]$$

CHAPTER V.

OBLIQUE-ANGLED TRIANGLES.

57. In the present Chapter we propose to consider the different cases which occur in the solution of oblique-angled triangles. As in right-angled triangles (Chap. IV.) there are six distinct cases, when we are given three of the parts, and are required to determine the remaining three.

Before proceeding to the general solutions of the triangles, we may notice that from particular values of the given parts the values of the other parts may be made to depend on the solutions of right-angled triangles, as for example—

α. Let the side a be a quadrant.

Here the supplemental triangle is right-angled; therefore, &c.

β Let the sides a and b be equal.

The triangle is divided into two equal right-angled triangles by the perpendicular to the base passing through the vertex; therefore, &c.

γ. The general solution of the equilateral triangle may be derived by aid of the *logarithmic* equation

$$\frac{\tan \frac{1}{2} a}{\tan a} = \cos A, \text{ or } 2 \cos A = \sec^2 \tfrac{1}{2} a.$$

58. The six cases (Art. 57) which present themselves are the following :—

I. Having given three sides, a, b, c;
II. Having given three angles, A, B, C;
III. Having given two sides and the included angle, a, C, b;
IV. Having given two angles and the adjacent side, A, c, B;
V. Having given two sides and an angle opposite either, a, b, A;
VI. Having given two angles and a side opposite either, A, B, a;

but these are immediately resolved into *three* distinct problems by the aid of the polar triangle.

For, when three sides are given, and the angles of the triangle are required, the data applied to the polar triangle transforms the problem into a supplemental problem, viz.—Having given three angles of a triangle, find the sides.

Similarly, Cases III. and IV. are supplemental, likewise V. and VI.

59. When numerical values are assigned to the given parts, in order to ascertain the remaining parts in degrees, minutes, and seconds, we employ formulæ adapted to logarithmic computation.

All the formulæ given in Chap. III. are either in logarithmic form or can immediately be thrown into the desired shape by means of a *subsidiary angle.* For the present we select the logarithmic formulæ, and apply

them to the following numerical examples, while we reserve the remaining formulæ for future discussion. (See Art. 76.)

60. Case I.—*Having given the three sides a, b, c.* Here we have (Art 33),

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}$$

$$= \frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

The part under the radical, being a symmetric function of the sides, appears on the right-hand side of the equations for determining A, B, C, and when once calculated is utilized in the calculation of each angle.

The logarithmic form of the above equation is

$$L\tan\frac{A}{2} = 10 - L\sin(s-a) + \tfrac{1}{2}\{L\sin(s-a) + L\sin(s-b) + L\sin(s-c) - L\sin s\}.$$

Examples.

(1). Given $a = 46° 24'$, $b = 67° 14'$, $c = 81° 12'$.

$L\sin s = L\sin 97° 25' = 9{\cdot}9963513.$

$L\sin(s-a) = L\sin 51° 1' = 9{\cdot}8906049.$

$L\sin(s-b) = L\sin 30° 11' = 9{\cdot}7013681.$

$L\sin(s-c) = L\sin 16° 13' = 9{\cdot}4460251.$

$L\tan\tfrac{1}{2}A = 9{\cdot}6302185$; $\therefore A = 46° 13' 30''.$

$L\tan\tfrac{1}{2}B = 9{\cdot}8194553$; $\therefore B = 66° 50' 20''.$

$L\tan\tfrac{1}{2}C = 10{\cdot}0747984$. $\therefore C = 99° 49' 10''.$

(2). Given $a = 108^\circ\ 14'$, $b = 75^\circ\ 29'$, $c = 56^\circ\ 37'$.

$$
\begin{aligned}
L \sin s &= L \sin 120^\circ\ 10' = 9\cdot9367988. \\
L \sin (s - a) &= L \sin\ \ 11^\circ\ 56' = 9\cdot3154947. \\
L \sin (s - b) &= L \sin\ \ 44^\circ\ 41' = 9\cdot8470714. \\
L \sin (s - c) &= L \sin\ \ 63^\circ\ 33' = 9\cdot9519799.
\end{aligned}
$$

$$
\begin{aligned}
L \tan \tfrac{1}{2} A &= 10\cdot2733789; \quad \therefore A = 123^\circ\ 53'\ 47''. \\
L \tan \tfrac{1}{2} B &= 9\cdot7418022; \quad \therefore B = 57^\circ\ 46'\ 56''. \\
L \tan \tfrac{1}{2} C &= 9\cdot6368937; \quad \therefore C = 46^\circ\ 51'\ 51\cdot5''.
\end{aligned}
$$

(3). Given $a = 57^\circ\ 17'$, $b = 20^\circ\ 39'$, $c = 76^\circ\ 22'$.

$$
\begin{aligned}
L \sin s &= L \sin 77^\circ\ \ 9' = 9\cdot9889849. \\
L \sin (s - a) &= L \sin 19^\circ\ 52' = 9\cdot5312649. \\
L \sin (s - b) &= L \sin 56^\circ\ 30' = 9\cdot9211066. \\
L \sin (s - c) &= L \sin\ \ 0^\circ\ 47' = 8\cdot1358104.
\end{aligned}
$$

$$
\begin{aligned}
L \tan \tfrac{1}{2} A &= 9\cdot2683336; \quad \therefore A = 21^\circ\ \ 1'\ \ 2''. \\
L \tan \tfrac{1}{2} B &= 8\cdot8784919; \quad \therefore B = 8^\circ\ 38'\ 46''. \\
L \tan \tfrac{1}{2} C &= 10\cdot6637881; \quad \therefore C = 155^\circ\ 31'\ 36\cdot5''.
\end{aligned}
$$

(4). Given $a = 68^\circ\ 45'$, $b = 53^\circ\ 15'$, $c = 46^\circ\ 30'$.

$$
\begin{aligned}
L \sin s &= L \sin 84^\circ\ 15' = 9\cdot9978093. \\
L \sin (s - a) &= L \sin 15^\circ\ 30' = 9\cdot4268988. \\
L \sin (s - b) &= L \sin 31^\circ\ \ 0' = 9\cdot7118393. \\
L \sin (s - c) &= L \sin 37^\circ\ 45' = 9\cdot7869056.
\end{aligned}
$$

$$
\begin{aligned}
L \tan \tfrac{1}{2} A &= 10\cdot0370184; \quad \therefore A = 94^\circ\ 52'\ 40''. \\
L \tan \tfrac{1}{2} B &= 9\cdot7520779; \quad \therefore B = 58^\circ\ 5\ \ '\ 10''. \\
L \tan \tfrac{1}{2} C &= 9\cdot6770116; \quad \therefore C = 50^\circ\ 50'\ 52\tfrac{1}{2}''.
\end{aligned}
$$

(5). Given $a = 63^\circ\ 54'$, $b = 47^\circ\ 18'$, $c = 53^\circ\ 26'$.

$$
\begin{aligned}
L \sin s &= L \sin 82^\circ\ 19' = 9\cdot9960834. \\
L \sin (s - a) &= L \sin 18^\circ\ 25' = 9\cdot4995840. \\
L \sin (s - b) &= L \sin 35^\circ\ \ 1' = 9\cdot7587717. \\
L \sin (s - c) &= L \sin 28^\circ\ 53' = 9\cdot6839720.
\end{aligned}
$$

$$L \tan \tfrac{1}{2} A = 9{\cdot}9735382; \therefore A = 86^\circ\ 30'\ 40''.$$
$$L \tan \tfrac{1}{2} B = 9{\cdot}7143505; \therefore B = 54^\circ\ 46'\ 14''.$$
$$L \tan \tfrac{1}{2} C = 9{\cdot}7891502; \therefore C = 63^\circ\ 12'\ 55\tfrac{1}{2}''.$$

61. Case II.—*Having given the three angles A, B, C.* Here we have (Art. 36),

$$\tan\frac{a}{2} = \sqrt{\frac{-\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}}$$

$$= \cos(S-A)\sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}},$$

with similar values for $\tan\frac{b}{2}$ and $\tan\frac{c}{2}$.

Examples.

(1). Given $A = 68^\circ\ 30'$, $B = 74^\circ\ 20'$, $C = 83^\circ\ 10'$.

$$L(-\cos S) = L \cos 67^\circ\ 0' = 9{\cdot}5918780.$$
$$L \cos(S-A) = L \cos 44^\circ\ 30' = 9{\cdot}8532421.$$
$$L \cos(S-B) = L \cos 38^\circ\ 40' = 9{\cdot}8925365.$$
$$L \cos(S-C) = L \cos 29^\circ\ 50' = 9{\cdot}9382576.$$

$$L \tan\frac{a}{2} = 9{\cdot}8071630; \therefore a = 65^\circ\ 21'\ 22\tfrac{1}{2}''$$
$$L \tan\frac{b}{2} = 9{\cdot}8464574; \therefore b = 70^\circ\ 9'\ 9\tfrac{1}{2}$$
$$L \tan\frac{c}{2} = 9{\cdot}8921785; \therefore c = 75^\circ\ 55'\ 9''.$$

(2). Given $A = 86^\circ\ 20'$, $B = 76^\circ\ 30'$, $C = 94^\circ\ 40'$.

$$L(-\cos S) = L \cos 51^\circ\ 15' = 9{\cdot}7965212$$
$$L \cos(S-A) = L \cos 42^\circ\ 25' = 9{\cdot}8682088.$$
$$L \cos(S-B) = L \cos 52^\circ\ 15' = 9{\cdot}7869056.$$
$$L \cos(S-C) = L \cos 34^\circ\ 5' = 9{\cdot}9181475.$$

$\tan \frac{1}{2} a = 9{\cdot}9798385$; $\therefore a = 87° 20' 28''$.

$\tan \frac{1}{2} b = 9{\cdot}8985353$; $\therefore b = 76° 44' 2\frac{1}{2}''$.

$\tan \frac{1}{2} c = 10{\cdot}0297772$; $\therefore c = 93° 55' 31''$.

(3). Given $A = 96° 45'$, $B = 108° 30'$, $C = 116° 15'$.

$L(-\cos S) = L \cos 19° 45' = 9{\cdot}9750129$.

$L \cos (S - A) = L \cos 61° \ 0' = 9{\cdot}6418420$.

$L \cos (S - B) = L \cos 52° 15' = 9{\cdot}7869056$.

$L \cos (S - C) = L \cos 44° 30' = 9{\cdot}9532421$.

$\tan \frac{1}{2} a = 9{\cdot}9883536$; $\therefore a = 88° 27' 49''$.

$\tan \frac{1}{2} b = 10{\cdot}1334172$; $\therefore b = 107° 19' 52''$.

$\tan \frac{1}{2} c = 10{\cdot}1997537$; $\therefore c = 115° 28' 13\frac{1}{4}''$.

(4). Given $A = 78° 30'$, $B = 118° 40'$, $C = 93° 20'$.

$L(-\cos S) = L \cos 34° 45' = 9{\cdot}9146852$.

$L \cos (S - A) = L \cos 66° 45' = 9{\cdot}5963154$.

$L \cos (S - B) = L \cos 26° 35' = 9{\cdot}9514757$.

$L \cos (S = C) = L \cos 51° 55' = 9{\cdot}7901493$.

$\tan \frac{1}{2} a = 9{\cdot}8846878$; $\therefore a = 74° 57' 46''$.

$\tan \frac{1}{2} b = 10{\cdot}2398481$; $\therefore b = 120° \ 8' 49''$

$\tan \frac{1}{2} c = 10{\cdot}0785217$; $\therefore c = 100° 18' 11\frac{3}{4}''$.

(5). Given $A = 57° 50'$, $B = 98° 20'$, $C = 63° 40'$.

$L(-\cos S) = L \cos 70° \ 5' = 9{\cdot}5323123$.

$L \cos (S - A) = L \cos 52° \ 5' = 9{\cdot}7885323$.

$L \cos (S - B) \quad L \cos 11° 35' = 9{\cdot}9910637$.

$L \cos (S - C) = L \cos 46° 15' = 9{\cdot}8398004$.

$\tan \frac{1}{2} a = 9{\cdot}7449903$; $\therefore a = 58° 8' 19''$.

$\tan \frac{1}{2} b = 9{\cdot}9475217$; $\therefore b = 83° 5' 36''$.

$\tan \frac{1}{2} c = 9{\cdot}7962584$; $\therefore c = 64° 3' 20''$.

62. Case III.—*Having given two sides and the included angle, a, C, b.*

This case is somewhat similar to the corresponding one in *Plane Trig.*, the base angles being calculated from two separate equations, one giving half the sum and the other half the difference of those angles. The formulæ to be employed have been arrived at on Art. 32 (Note to Ex. 8). They have, moreover, been demonstrated geometrically from the relations between the sides and angles of certain right-angled triangles. As they are of fundamental importance in the solution of triangles, we shall now proceed by a direct method to obtain the same results.

63. **Napier's Analogies:**

Let
$$x = \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b};$$
hence,
$$x = \frac{\sin A \pm \sin B}{\sin a \pm \sin b}. \tag{1}$$
Now,
$$\cos A + \cos B \cos C = \sin B \sin C \cos a = x \sin C \sin b \cos a,$$
and
$$\cos B + \cos C \cos A = \sin C \sin A \cos b = x \sin C \sin a \cos b.$$

By addition,
$$(\cos A + \cos B)(1 + \cos C) = x \sin C \sin (a + b); \tag{2}$$
by subtraction,
$$(\cos B - \cos A)(1 - \cos C) = x \sin C \, (a - b). \tag{3}$$

On taking the positive sign in (1), and dividing by (2), we obtain

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin a + \sin b}{\sin (a + b)} \cdot \frac{1 + \cos C}{\sin C};$$

or

$$\tan \tfrac{1}{2} (A + B) = \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)} \cot \frac{C}{2}. \qquad (\alpha')$$

On taking the negative sign in (1), we obtain, by a similar method,

$$\tan \tfrac{1}{2} (A - B) = \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)} \cot \frac{C}{2}. \qquad (\beta')$$

Employing the supplemental formulæ, viz.,

$$\cos a - \cos b \cos c = \sin b \sin c \cos A = \frac{1}{x} \sin B \cos A \sin c,$$

and

$$\cos b - \cos c \cos a = \sin c \sin a \cos B = \frac{1}{x} \sin A \cos B \sin c,$$

we have, as before,

$$x \,.\, (\cos a \pm \cos b)(1 \mp \cos c) = \sin c \sin (B \pm A).$$

Dividing each of these equations by results in equation (1) to eliminate x, we have formulæ supplemental to (α') and (β'), viz.,

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \frac{c}{2}, \qquad (\gamma')$$

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \frac{c}{2}. \qquad (\delta')$$

64. Delambre's (or Gauss's) Analogies:

By the preceding Article, we have

$$\tan \tfrac{1}{2}(A + B) \tan \frac{C}{2} = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}.$$

Hence,

$$1 - \tan \tfrac{1}{2}(A + B) \tan \tfrac{1}{2} C = 1 - \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)};$$

or

$$\frac{-\cos S}{\cos \frac{1}{2}(A + B) \cos \frac{1}{2} C} = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2}(a + b)}; \qquad (1)$$

but

$$-\cos S = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b \sin C}{\cos \frac{1}{2} c}.$$

[Art. 43, Ex. 28.]

Substituting this value in (1), we have at once

$$\cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C. \qquad (a)$$

Again,

$$1 + \tan \tfrac{1}{2}(A - B) \tan \tfrac{1}{2} C = 1 + \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}.$$

Hence,

$$\frac{\cos (S - A)}{\cos \frac{1}{2}(A - B) \cos \frac{1}{2} C} = \frac{2 \sin \frac{1}{2} a \cos \frac{1}{2} b}{\sin \frac{1}{2}(a + b)}; \qquad (2)$$

but (Art. 43, Ex. 31)

$$\cos (S - A) = \frac{\cos \frac{1}{2} b \cos \frac{1}{2} c}{\cos \frac{1}{2} a} \sin A = \sin \tfrac{1}{2} a \cos \tfrac{1}{2} c \frac{\sin B}{\sin \frac{1}{2} b}.$$

Substituting this value in (2), we have

$$\cos \tfrac{1}{2}(A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C. \qquad (\beta)$$

Similarly,

$$\sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a - b) \cos \tfrac{1}{2} C, \qquad (\gamma)$$

and

$$\sin \tfrac{1}{2}(A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a - b) \cos \tfrac{1}{2} C. \qquad (\delta)$$

65. It will be thus seen that when each of Napier's Analogies is added to or subtracted from unity, the results are easily transformed into Delambre's formulæ.

It is also to be noticed that if the equation

$$\cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C$$

be applied to the colunar triangle of parts a, $\pi - b$, $\pi - c$, A, $\pi - B$, $\pi - C$, we get

$$\sin \tfrac{1}{2}(A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a - b) \cos \tfrac{1}{2} C.$$

And by a similar method, the equation

$$\sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a - b) \cos \tfrac{1}{2} C$$

gives

$$\cos \tfrac{1}{2}(A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C.$$

Remark.—From the three preceding Articles it has been shown that Napier's four Analogies are reducible, by aid of the polar triangle, to *two independent theorems*, involving five parts of a spherical triangle; and that Gauss's four formulæ are, by aid of the colunar triangle, likewise reducible to two independent theorems.

66. In the solution of triangles, CASE III., the formulæ

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C,$$

and

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C,$$

enable us to calculate $\frac{1}{2}(A + B)$, and $\frac{1}{2}(A - B)$, from which we obtain A and B.

The remaining element c is determined from the equation

$$\cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C;$$

or, as will be hereafter shown, it may be determined from the formula

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

without having previously determined A or B.

Examples.

(1). Given $a = 43° 18'$, $b = 19° 24'$, $C = 74° 22'$.
Firstly, determine A and B—

$$L \cos \tfrac{1}{2}(a - b) = L \cos 11° 57' = 9{\cdot}9904848.$$

$$L \cos \tfrac{1}{2}(a + b) = L \cos 31° 21' = 9{\cdot}9314605.$$

$$L \cot \tfrac{1}{2} C \quad = L \cot 37° 11' = 10{\cdot}1199969.$$

$$L \tan \tfrac{1}{2}(A + B) = 10{\cdot}1790212.$$

$$L \sin \tfrac{1}{2}(a - b) = 9{\cdot}3160921.$$

$$L \sin \tfrac{1}{2}(a + b) = 9{\cdot}7162243.$$

$$L \tan \tfrac{1}{2}(A - B) = 9{\cdot}7198647.$$

Therefore,

$$\tfrac{1}{2}(A + B) = 56° 29' 17'', \quad \text{and} \quad \tfrac{1}{2}(A - B) = 27° 41' 0\tfrac{1}{2}''.$$

To determine c—

$$L\cos\tfrac{1}{2}(a+b) = L\cos 31^\circ\ 21'\ \ 0'' = 9{\cdot}9314605.$$
$$L\cos\tfrac{1}{2}(A+B) = L\cos 56^\circ\ 29'\ 17'' = 9{\cdot}7420263.$$
$$L\sin\tfrac{1}{2}C = L\sin 37^\circ\ 11'\ \ 0'' = 9{\cdot}7813010.$$

Hence

$$L\cos\tfrac{1}{2}c = 9{\cdot}9707352, \quad \text{or} \quad \tfrac{1}{2}c = 20^\circ\ 48'\ 54\tfrac{1}{4}''.$$

Ans. $A = 84^\circ\ 10'\ 17\tfrac{1}{2}''$, $B = 28^\circ\ 48'\ 17\tfrac{1}{2}''$, $c = 41^\circ\ 35'\ 48\tfrac{1}{2}''$.

(2). $a = 96^\circ\ 24'\ 30''$, $b = 68^\circ\ 27'\ 26''$, $C = 84^\circ\ 46'\ 40'$.

$$L\cos\tfrac{1}{2}(a-b) = L\cos 13^\circ\ 58'\ 32'' = 9{\cdot}9869503.$$
$$L\cos\tfrac{1}{2}(a+b) = L\cos 82^\circ\ 25'\ 58'' = 9{\cdot}1195505.$$
$$L\cot\tfrac{1}{2}C = L\cot 42^\circ\ 23'\ 20'' = 10{\cdot}0396387.$$
$$L\tan\tfrac{1}{2}(A+B) = 10{\cdot}9070385.$$
$$L\sin\tfrac{1}{2}(a-b) = 9{\cdot}3829313, \quad L\sin\tfrac{1}{2}(a+b) = 9.9962011.$$
$$L\tan\tfrac{1}{2}(A-B) = 9{\cdot}4263689.$$
$$\tfrac{1}{2}(A+B) = 82^\circ\ 56'\ 19\tfrac{3}{4}'', \quad \tfrac{1}{2}(A-B) = 14^\circ\ 56'\ 40\tfrac{1}{2}''.$$
$$L\cos\tfrac{1}{2}(A+B) = 9{\cdot}0896545.$$
$$L\sin\tfrac{1}{2}C = 9{\cdot}8287624.$$
$$L\cos\tfrac{1}{2}c = 9{\cdot}8586584, \quad \tfrac{1}{2}c = 43^\circ\ 45'\ 48\tfrac{1}{2}''.$$

Ans. $A = 97^\circ\ 53'\ 0\tfrac{1}{4}''$, $B = 67^\circ\ 59'\ 39\tfrac{1}{4}''$, $c = 87^\circ\ 31'\ 37''$.

(3). $a = 76^\circ\ 24'\ 40''$, $b = 58^\circ\ 18'\ 36''$, $C = 116^\circ\ 30'\ 28''$.

$$L\cos\tfrac{1}{2}(a-b) = L\cos 9^\circ\ \ 3'\ \ 2'' = 9{\cdot}9945590.$$
$$L\cos\tfrac{1}{2}(a+b) = L\cos 67^\circ\ 21'\ 38'' = 9{\cdot}5853827.$$
$$L\cot\tfrac{1}{2}C = L\cot 58^\circ\ 15'\ 14'' = 9{\cdot}7914976.$$
$$L\tan\tfrac{1}{2}(A+B) = 10{\cdot}2006739.$$
$$L\sin\tfrac{1}{2}(a-b) = 9{\cdot}1967450, \quad L\sin\tfrac{1}{2}(a+b) = 9{\cdot}9651760.$$
$$L\tan\tfrac{1}{2}(A-B)\ 9{\cdot}0230666.$$
$$\tfrac{1}{2}(A+B) = 57^\circ\ 47'\ 23\tfrac{3}{4}'', \quad \tfrac{1}{2}(A-B) = 6^\circ\ 1'\ 11\tfrac{1}{2}''.$$

$$L \cos \tfrac{1}{2}(A+B) \quad 9\cdot7267476.$$

$$L \sin \tfrac{1}{2} C \quad = 9\cdot9296171.$$

$$\tfrac{1}{2} c = 52^\circ\ 6'\ 43\tfrac{1}{2}''.$$

Ans. $A = 63^\circ\ 48'\ 35\tfrac{1}{4}''$, $B = 51^\circ\ 46'\ 12\tfrac{1}{4}''$, $c = 104^\circ\ 13'\ 27''$.

(4). $a = 86^\circ\ 18'\ 40''$, $b = 45^\circ\ 36'\ 20''$, $C = 120^\circ\ 46'\ 30''$.

$$L \cos \tfrac{1}{2}(a-b) = L \cos 20^\circ\ 21'\ 10'' = 9\cdot9720032.$$

$$L \cos \tfrac{1}{2}(a+b) = L \cos 65^\circ\ 57'\ 30'' = 9\cdot6100219.$$

$$L \cot \tfrac{1}{2} C \quad = L \cot 60^\circ\ 23'\ 15'' = 9\cdot7546296.$$

$$\tfrac{1}{2}(A+B) = 52^\circ\ 36'\ 4\tfrac{3}{4}''.$$

$$L \sin \tfrac{1}{2}(a-b) = 9\cdot5413288, \quad L \sin \tfrac{1}{2}(a+b) = 9\cdot9605894.$$

$$\tfrac{1}{2}(A-B) = 12^\circ\ 12'\ 49''.$$

$$L \sin \tfrac{1}{2} C \quad = 9\cdot9392132.$$

$$L \cos \tfrac{1}{2}(A+B) = 9\cdot7834444.$$

$$\tfrac{1}{2} c = 54^\circ\ 19'\ 35\tfrac{3}{4}''.$$

Ans. $A = 64^\circ\ 48'\ 53\tfrac{3}{4}''$, $B = 40^\circ\ 23'\ 15\tfrac{3}{4}''$, $c = 108^\circ\ 39'\ 11\tfrac{1}{2}''$.

(5). $a = 88^\circ\ 24'$, $b = 56^\circ\ 48'$, $C = 128^\circ\ 16'$.

$$L \cos \tfrac{1}{2}(a-b) = L \cos 15^\circ\ 48' = 9\cdot9832735.$$

$$L \cos \tfrac{1}{2}(a+b) = L \cos 72^\circ\ 36' = 9\cdot4757304.$$

$$L \cot \tfrac{1}{2} C \quad = L \cot 64^\circ\ 8' = 9\cdot6856120.$$

$$\tfrac{1}{2}(A+B) = 57^\circ\ 20'\ 27\tfrac{1}{4}''.$$

$$L \sin \tfrac{1}{2}(a-b) = 9\cdot4350161, \quad L \sin \tfrac{1}{2}(a+b) = 9\cdot9796578.$$

$$\tfrac{1}{2}(A-B) = 7^\circ\ 52'\ 36\tfrac{1}{4}''.$$

$$L \cos \tfrac{1}{2}(A+B) = 9\cdot7321029.$$

$$L \sin \tfrac{1}{2} C \quad = 9\cdot9541517.$$

$$\tfrac{1}{2} c = 60^\circ\ 5'\ 26''.$$

Ans. $A = 65^\circ\ 13'\ 3\tfrac{1}{2}''$, $B = 49^\circ\ 27'\ 51''$. $c = 120^\circ\ 10'\ 52''$.

(6). $a = b = 32° 23' 57''$, $C = 66° 49' 17''$.

—(*Science and Art Exam. Papers.*)

$L \cos \frac{1}{2}(a - b) = 10.$

$L \cos \frac{1}{2}(a + b) = L \cos 32° 23' 57'' = 9{\cdot}9265152.$

$L \cot \frac{1}{2} C = L \cot 33° 24' 38\frac{1}{2}'' = 10{\cdot}1806890.$

$\frac{1}{2}(A + B) = 60° 53' 2''.$

$L \sin a = 9{\cdot}7290144.$

$L \sin \frac{1}{2} C = 9{\cdot}7408650.$

$\frac{1}{2} c = 17° 9' 35\frac{1}{2}''.$

Ans. $A = B = 60° 53' 2''$, $c = 34° 19' 11''$.

67. Case IV.—*Having given two angles and the adjacent side* (A, c, B).

By *Napier's Analogies :*

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \tfrac{1}{2} c,$$

and

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \tfrac{1}{2} c;$$

from which we may determine $\frac{1}{2}(a + b)$ and $\frac{1}{2}(a - b)$, and thence a and b.

The remaining part C is determined by the equation

$$\cos \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C = \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c.$$

It is evident that the value of C thus found is ambiguous,* inasmuch as the angle is obtained from the *sine;* and the values are, therefore, C and $180 - C$. The equation has, however, been selected as an example, to show that an

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* By using formula γ (or δ), Art. 64, ambiguity is avoided

ambiguity may sometimes be got rid of by attending to the relations which the parts of a triangle bear to each other; and in the following numerical examples the ambiguity is removed by remembering that the greater side is opposite to the greater angle.

Examples.

(1). $A = 68° \, 40'$, $B = 56° \, 20'$, $c = 84° \, 30'$.

$$L \cos \tfrac{1}{2}(A - B) = L \cos \; 6° \, 10' = 9 \cdot 9974797.$$
$$L \cos \tfrac{1}{2}(A + B) = L \cos 62° \, 30' = 9 \cdot 6644056.$$
$$L \tan \tfrac{1}{2} c \qquad = L \tan 42° \, 15' = 9 \cdot 9582465.$$
$$\tfrac{1}{2}(a + b) = 62° \, 55' \, 9''.$$
$$L \sin \tfrac{1}{2}(A - B) = 9 \cdot 0310890, \quad L \sin \tfrac{1}{2}(A + B) = 9 \cdot 9479289.$$
$$\tfrac{1}{2}(a - b) = 6° \, 16' \, 39''.$$
$$L \cos \tfrac{1}{2}(a + b) = 9 \cdot 6582472.$$
$$L \cos \tfrac{1}{2} c \qquad = 9 \cdot 8693597.$$
$$\tfrac{1}{2} C = 48° \, 39' \, 31\tfrac{3}{4}''.$$

Ans. $a = 60° \, 11' \, 48''$, $b = 56° \, 38' \, 30''$, $C = 97° \, 19' \, 3\tfrac{1}{2}''$

(2). $A = 31° \, 34' \, 26''$, $B = 30° \, 28' \, 12''$, $c = 70° \, 2' \, 3''$.

$$L \cos \tfrac{1}{2}(A - B) = L \cos \; 0° \, 33' \; 7'' = 9 \cdot 9999799.$$
$$L \cos \tfrac{1}{2}(A + B) = L \cos 31° \; 1' \, 19'' = 9 \cdot 9329656.$$
$$L \tan \tfrac{1}{2} c \qquad = L \tan 35° \; 1' \, 1\tfrac{1}{2}'' = 9 \cdot 8455023.$$
$$\tfrac{1}{2}(a + b) = 39° \, 16' \, 4\tfrac{1}{4}''.$$
$$L \sin \tfrac{1}{2}(A - B) = 7 \cdot 9837459, \quad L \sin \tfrac{1}{2}(A + B) = 9 \cdot 7121160.$$
$$\tfrac{1}{2}(a - b) = 0° \, 45' \, 1''.$$
$$L \cos \tfrac{1}{2}(a + b) = 9 \cdot 8888507.$$
$$L \cos \tfrac{1}{2} c \qquad = 9 \cdot 9132738.$$
$$\tfrac{1}{2} C = 65° \, 1' \, 55''.$$

Ans. $a = 40° \, 1' \, 5\tfrac{1}{4}''$, $b = 38° \, 31' \, 3\tfrac{1}{4}''$, $C = 130° \, 3' \, 50''$.

(3). $A = 130° \; 5' \; 22{\cdot}41''$, $B = 32° \; 26' \; 6{\cdot}41''$, $c = 51° \; 6' \; 11{\cdot}6''$.

—(*Science Siz. Exam. Papers.*)

$$L \cos \tfrac{1}{2}(A - B) = L \cos 48° \; 49' \; 38'' \quad = 9{\cdot}8184449.$$

$$L \cos \tfrac{1}{2}(A + B) = L \cos 81° \; 15' \; 44{\cdot}41'' = 9{\cdot}1815879.$$

$$L \tan \tfrac{1}{2} c \quad = L \tan 25° \; 33' \; 5{\cdot}8'' \quad = 9{\cdot}6795022.$$

$$\tfrac{1}{2}(a + b) = 64° \; 14' \; 7''.$$

$$L \sin \tfrac{1}{2}(A - B) = 9{\cdot}8766380, \quad L \sin \tfrac{1}{2}(A + B) = 9{\cdot}9949301.$$

$$\tfrac{1}{2}(a - b) = 20° \; 0' \; 22''.$$

$$L \cos \tfrac{1}{2}(a + b) = 9{\cdot}6381663.$$

$$L \cos \tfrac{1}{2} c \quad = 9{\cdot}9552944.$$

$$\tfrac{1}{2} C = 18° \; 22' \; 43''.$$

Ans. $a = 84° \; 14' \; 29''$, $b = 44° \; 13' \; 45''$, $C = 36° \; 45' \; 26''$.

(4). $A = 96° \; 46' \; 30''$, $B = 84° \; 30' \; 20''$, $c = 126° \; 46'$.

$$L \cos \tfrac{1}{2}(A - B) \quad = L \cos \; 6° \; 8' \; 5'' = 9{\cdot}9975058.$$

$$*L \cos \text{supplement of } \tfrac{1}{2}(A + B) = L \cos 89° \; 21' \; 35'' = 8{\cdot}0482011.$$

$$L \tan \tfrac{1}{2} c \quad = L \tan 63° \; 23' \; 0'' = 10{\cdot}3000526.$$

$$\text{Supplement of } \tfrac{1}{2}(a + b) = 89° \; 40' \; 38''.$$

$$L \sin \tfrac{1}{2}(A - B) = 9{\cdot}0288420, \quad L \sin \tfrac{1}{2}(A + B) = 9{\cdot}9999729.$$

$$\tfrac{1}{2}(a - b) = 12° \; 2' \; 20''.$$

$$L \cos \text{supplement of } \tfrac{1}{2}(a + b) = 7{\cdot}7506455, \quad L \cos \tfrac{1}{2} c = 9{\cdot}6512966$$

$$\tfrac{1}{2} C = 62° \; 44' \; 6\tfrac{5}{8}''.$$

Ans. $a = 102° \; 21' \; 42''$, $b = 78° \; 17' \; 2''$, $C = 125° \; 28' \; 13\tfrac{1}{4}''$.

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* When $\frac{1}{2}(A + B)$ exceeds a quadrant, by substituting its supplement in the equation, a value is obtained for the supplement of $\frac{1}{2}(a + b)$. See *Napier's Analogies*, and *Gauss's Formulæ*, from which it also appears that $\frac{1}{2}(A + B)$ and $\frac{1}{2}(a + b)$ are of the same affection.

(5). $A = 84° 30' 20''$, $B = 76° 20' 40''$, $c = 130° 46'$.

$$L \cos \tfrac{1}{2} (A - B) = L \cos 4° \; 4' \; 50'' = 9{\cdot}9988977.$$
$$L \cos \tfrac{1}{2} (A + B) = L \cos 80° \; 25' \; 30'' = 9{\cdot}2209927.$$
$$L \tan \tfrac{1}{2} c \qquad = L \tan 65° \; 23' \; 0'' = 10{\cdot}3389566.$$
$$\tfrac{1}{2} (a + b) = 85° \; 37' \; 50\tfrac{1}{4}''.$$
$$L \sin \tfrac{1}{2} (A - B) = 8{\cdot}8522289, \quad L \sin \tfrac{1}{2} (A + B) = 9{\cdot}9939071.$$
$$\tfrac{1}{2} (a - b) = 8° \; 57' \; 2''.$$
$$L \cos \tfrac{1}{2} (a + b) = 8{\cdot}8818753.$$
$$L \cos \tfrac{1}{2} c \qquad = 9{\cdot}6196622.$$
$$\tfrac{1}{2} C = 65° \; 25' \; 46\tfrac{3}{4}''.$$

Ans. $a = 94° \; 34' \; 52\tfrac{1}{4}''$, $b = 76° \; 40' \; 48\tfrac{1}{4}''$, $C = 130° \; 51' \; 33\tfrac{1}{2}''$.

68. Case V.—*Having given two sides and the angle opposite either* (a, b, A).

The angle B is found from the formula

$$\sin B = \frac{\sin b}{\sin a} \sin A.$$

For the angle C,

$$\tan \tfrac{1}{2} C = \frac{\cos \tfrac{1}{2} (a - b)}{\cos \tfrac{1}{2} (a + b)} \cot \tfrac{1}{2} (A + B).$$

For the side c,

$$\tan \tfrac{1}{2} c = \frac{\cos \tfrac{1}{2} (A + B)}{\cos \tfrac{1}{2} (A - B)} \tan \tfrac{1}{2} (a + b).$$

Since B has been found from its sine, it may have two values, viz. B and $180 - B$, and the triangle in the general case will admit of two solutions.

This is known as the *ambiguous case*, a full discussion of which is given in Arts. 70–75.

The following numerical examples will serve to elucidate how two different triangles may be constructed when three parts are given :—

Examples.

(1). $a = 42° 45'$, $b = 47° 15'$, $A = 56° 30'$.

$$L \sin a = L \sin 42° 45' = 9{\cdot}8317423.$$

$$L \sin b = L \sin 47° 15' = 9{\cdot}8658868.$$

$$L \sin A = L \sin 56° 30' = 9{\cdot}9211066.$$

Hence

$$L \sin B = 9{\cdot}9552511 ;$$

or

$$B = 64° 26' 4'', \quad \text{or} \quad 115° 33' 56''.$$

Firstly—take the value of B less than a right angle.

$$L \cos \tfrac{1}{2}(b + a) = L \cos 45° \ 0' \ 0'' = 9{\cdot}8494850.$$

$$L \cos \tfrac{1}{2}(b - a) = L \cos \ 2° 15' \ 0'' = 9{\cdot}9996650;$$

and

$$L \cot \tfrac{1}{2}(B + A) = L \cot 60° 28' 2'' = 9{\cdot}7532216.$$

Hence

$$L \tan \tfrac{1}{2} C = 9{\cdot}9034016;$$

or

$$\tfrac{1}{2} C = 38° 40' 47\tfrac{1}{2}''.$$

$$L \cos \tfrac{1}{2}(B + A) = L \cos \ 60° 28' \ 2'' = 9{\cdot}6927772.$$

$$L \cos \tfrac{1}{2}(B - A) = L \cos \ 3° 58' \ 2'' = 9{\cdot}9989581.$$

$$L \tan \tfrac{1}{2}(b + a) = L \tan 45° \ 0' \ 0'' = 10{\cdot}0000000.$$

$$\tfrac{1}{2} c = 26° 17' 39''.$$

Secondly—take the value of B greater than a right angle. Then

$$L \cot \tfrac{1}{2}(B + A) = L \cot 86° 1' 58'' = 8{\cdot}8410586,$$

and by aid of the logarithms in the first case we have

$$\tfrac{1}{2} C = 5° 35' 50\tfrac{1}{4}''.$$

Again,

$$L\cos\tfrac{1}{2}(B+A) = L\cos 86^\circ\ 1'\ 58'' = 8\cdot8400167.$$

$$L\cos\tfrac{1}{2}(B-A) = L\cos 29^\circ\ 31'\ 58'' = 9\cdot9395561.$$

Hence

$$\tfrac{1}{2}c = 4^\circ\ 32'\ 47\tfrac{1}{8}''.$$

Ans.—

$$B = 64^\circ\ 25'\ 4'',\quad C = 77^\circ\ 21'\ 35'',\quad c = 52^\circ\ 35'\ 18'';$$

$$B = 115^\circ\ 33'\ 56'',\quad C = 11^\circ\ 11'\ 40\tfrac{1}{2}''\quad c = 9^\circ\ 5'\ 34\tfrac{1}{4}''.$$

(2). $a = 62^\circ\ 15'\ 24''$, $b = 103^\circ\ 18'\ 47''$, $A = 53^\circ\ 42'\ 38''$.

$$L\sin a = L\sin 62^\circ\ 15'\ 24'' = 9\cdot9469638.$$

$$L\sin b = L\sin 76^\circ\ 41'\ 13'' = 9\cdot9881693.$$

$$L\sin A = L\sin 53^\circ\ 42'\ 38'' = 9\cdot9063552.$$

$$B = 62^\circ\ 24'\ 24\tfrac{4}{5}'',\quad\text{or}\quad 117^\circ\ 35'\ 35\tfrac{1}{5}''.$$

Firstly—

$$L\cos\tfrac{1}{2}(b+a) = L\cos 82^\circ\ 47'\ 5\tfrac{1}{2}'' = 9\cdot0989736.$$

$$L\cos\tfrac{1}{2}(b-a) = L\cos 20^\circ\ 31'\ 41\tfrac{1}{2}'' = 9\cdot9715077.$$

$$L\cot\tfrac{1}{2}(B+A) = L\cot 58^\circ\ 3'\ 31\tfrac{2}{5}'' = 9\cdot7947983.$$

$$\tfrac{1}{2}C = 77^\circ\ 51'\ 35\tfrac{2}{3}''.$$

$$L\cos\tfrac{1}{2}(B+A) = L\cos 58^\circ\ 3'\ 31\tfrac{2}{5}'' = 9\cdot7234966.$$

$$L\cos\tfrac{1}{2}(B-A) = L\cos 4^\circ\ 20'\ 53\tfrac{2}{5}' = 9\cdot9987482.$$

$$L\tan\tfrac{1}{2}(b+a) = L\tan 82^\circ\ 47'\ 5\tfrac{1}{2}'' = 10\cdot8975738.$$

$$\tfrac{1}{2}c = 76^\circ\ 34'\ 47\tfrac{3}{4}''.$$

Secondly—

$$L\cot\tfrac{1}{2}(B+A) = L\cot 85^\circ\ 39'\ 6\tfrac{3}{5}'' = 8\cdot8810181.$$

$$\tfrac{1}{2}C = 29^\circ\ 33'\ 5\tfrac{1}{3}''.$$

$$L\cos\tfrac{1}{2}(B+A) = L\cos 85^\circ\ 39'\ 6\tfrac{3}{5}'' = 8\cdot8797663.$$

$$L\cos\tfrac{1}{2}(B-A) = L\cos 31^\circ\ 56'\ 28\tfrac{3}{5}'' = 9\cdot9286983.$$

$$\tfrac{1}{2}c = 35^\circ\ 12'\ 43''.$$

Ans.—

$$B = 62^\circ\ 24'\ 24\tfrac{4}{5}'',\quad C = 155^\circ\ 43'\ 11\tfrac{1}{3}'',\quad c = 153^\circ\ 9'\ 35\tfrac{1}{2}'',$$

$$B = 117^\circ\ 35'\ 35\tfrac{1}{5}'',\quad C = 59^\circ\ 6'\ 10\tfrac{2}{3}'',\quad c = 70^\circ\ 25'\ 26''.$$

(3). $a = 52° 45' 20''$, $b = 71° 12' 40''$, $A = 46° 22' 10''$.

$$B = 59° 24' 15\tfrac{3}{4}'', \quad \text{or} \quad 120° 35' 44\tfrac{1}{4}''.$$

Firstly—

$$L \cos \tfrac{1}{2}(b + a) = L \cos 61° 59' 0'' = 9{\cdot}6718468.$$
$$L \cos \tfrac{1}{2}(b - a) = L \cos 9° 13' 40'' = 9{\cdot}9943430.$$
$$L \cot \tfrac{1}{2}(B + A) = L \cot 52° 53' 12\tfrac{7}{8}'' = 9{\cdot}8788970.$$
$$\tfrac{1}{2} C = 57° 49' 57\tfrac{3}{4}''.$$

$$L \cos \tfrac{1}{2}(B + A) = L \cos 52° 53' 12\tfrac{7}{8}'' = 9{\cdot}7805983.$$
$$L \cos \tfrac{1}{2}(B - A) = L \cos 6° 31' 2\tfrac{7}{8}'' = 9{\cdot}9971842.$$
$$L \tan \tfrac{1}{2}(b + a) = L \tan 61° 59' 0'' = 10{\cdot}2740209.$$
$$\tfrac{1}{2} c = 48° 46' 39\tfrac{2}{5}''.$$

Secondly—

$$L \cot \tfrac{1}{2}(B + A) = L \cot 83° 28' 57\tfrac{1}{8}'' = 9{\cdot}0578349.$$
$$\tfrac{1}{2} C = 13° 29' 57\tfrac{3}{5}''.$$

$$L \cos \tfrac{1}{2}(B + A) = L \cos 83° 28' 57\tfrac{1}{8}'' = 9{\cdot}0550190.$$
$$L \cos \tfrac{1}{2}(B - A) = L \cos 37° 6' 47\tfrac{1}{8}'' = 9{\cdot}9017013.$$
$$\tfrac{1}{2} c = 14° 58' 35\tfrac{1}{4}''.$$

Ans.—

$$B = 59° 24' 15\tfrac{3}{4}'', \quad C = 115° 39' 55\tfrac{1}{2}'', \quad c = 97° 33' 18\tfrac{4}{5}'',$$
$$B = 120° 35' 44\tfrac{1}{4}'', \quad C = 26° 59' 55\tfrac{1}{5}'', \quad c = 29° 57' 10\tfrac{1}{2}''.$$

(4) $a = 48° 45' 40''$, $b = 67° 12' 20''$, $A = 42° 20' 30''$.

$$B = 55° 39' 57'', \quad \text{or} \quad 124° 20' 3''.$$

Firstly—

$$L \cos \tfrac{1}{2}(b + a) = L \cos 57° 59' 0'' = 9{\cdot}7244118.$$
$$L \cos \tfrac{1}{2}(b - a) = L \cos 9° 13' 20'' = 9{\cdot}9943498.$$
$$L \cot \tfrac{1}{2}(B + A) = L \cot 49° 0' 13\tfrac{1}{2}'' = 9{\cdot}9391057.$$
$$\tfrac{1}{2} C = 58° 17' 9''.$$

$$L \cos \tfrac{1}{2}(B + A) = L \cos 49° 0' 13\tfrac{1}{2}'' = 9{\cdot}8169102.$$
$$L \cos \tfrac{1}{2}(B - A) = L \cos 6° 39' 43\tfrac{1}{2}'' = 9{\cdot}9970576.$$
$$L \tan \tfrac{1}{2}(b + a) = L \tan 57° 59' 0'' = 10{\cdot}2029297.$$

$$\tfrac{1}{2} c = 46° 34' 4\tfrac{4}{5}''.$$

Secondly—

$$L \cot \tfrac{1}{2}(B + A) = L \cot 83^\circ\ 20'\ 16\tfrac{1}{2}'' = 9\cdot0674506.$$

$$\tfrac{1}{2} C = 12^\circ\ 16'\ 7\tfrac{1}{2}''.$$

$$L \cos \tfrac{1}{2}(B + A) = L \cos 83^\circ\ 20'\ 16\tfrac{1}{2}'' = 9\cdot0645081.$$

$$L \cos \tfrac{1}{2}(B - A) = L \cos 40^\circ\ 59'\ 46\tfrac{1}{2}'' = 9\cdot8778046.$$

$$\tfrac{1}{2} c = 13^\circ\ 48'\ 40''.$$

Ans.—

$$B = 55^\circ\ 39'\ 57'', \quad C = 116^\circ\ 34'\ 18'', \quad c = 93^\circ\ 8'\ 9\tfrac{3}{8}'';$$

$$B = 124^\circ\ 20'\ 3'', \quad C = 24^\circ\ 32'\ 15'', \quad c = 27^\circ\ 37'\ 20''.$$

(5). $a = 46^\circ\ 20'\ 45''$, $b = 65^\circ\ 18'\ 15''$, $A = 40^\circ\ 10'\ 30''$.

$$B = 54^\circ\ 6'\ 19'', \quad \text{or} \quad 125^\circ\ 53'\ 41''.$$

Firstly—

$$L \cos \tfrac{1}{2}(b + a) = L \cos 55^\circ\ 49'\ 30'' = 9\cdot7495218.$$

$$L \cos \tfrac{1}{2}(b - a) = L \cos 9^\circ\ 28'\ 45'' = 9\cdot9940291.$$

$$L \cot \tfrac{1}{2}(B + A) = L \cot 47^\circ\ 8'\ 24\tfrac{1}{2}'' = 9\cdot9675258.$$

$$\tfrac{1}{2} C = 58^\circ\ 27'\ 43''.$$

$$L \cos \tfrac{1}{2}(B + A) = L \cos 47^\circ\ 8'\ 24\tfrac{1}{2}'' = 9\cdot8326414.$$

$$L \cos \tfrac{1}{2}(B - A) = L \cos 6^\circ\ 57'\ 54\tfrac{1}{2}'' = 9\cdot9967831.$$

$$L \tan \tfrac{1}{2}(b + a) = L \tan 55^\circ\ 49'\ 30'' = 10\cdot1681548.$$

$$\tfrac{1}{2} c = 45^\circ\ 15'\ 53''.$$

Secondly—

$$L \cot \tfrac{1}{2}(B + A) = L \cot 83^\circ\ 2'\ 5\tfrac{1}{2}'' = 9\cdot0869538.$$

$$\tfrac{1}{2} C = 12^\circ\ 6'\ 26\tfrac{2}{3}''.$$

$$L \cos \tfrac{1}{2}(B + A) = L \cos 83^\circ\ 2'\ 5\tfrac{1}{2}'' = 9\cdot0837368.$$

$$L \cos \tfrac{1}{2}(B - A) = L \cos 42^\circ\ 51'\ 35\tfrac{1}{2}'' = 9\cdot8651156.$$

$$\tfrac{1}{2} c = 13^\circ\ 41'\ 37''.$$

Ans.—

$$B = 54^\circ\ 6'\ 19'', \quad C = 116^\circ\ 55'\ 26'', \quad c = 90^\circ\ 31'\ 46''.$$

$$B = 125^\circ\ 53'\ 41'', \quad C = 24^\circ\ 12'\ 53\tfrac{1}{3}'', \quad c = 27^\circ\ 23'\ 14''.$$

69. Case VI.—*Having given two angles and a side opposite one of them* (A, B, a).

This Case reduces, by aid of the polar triangle, to the preceding, and the direct solution may therefore be obtained by means of formulæ supplemental to those employed in Art. 68, viz.,

$$\sin b = \frac{\sin B}{\sin A} \sin a,$$

$$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}(A + B),$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a + b).$$

The same ambiguities will thus present themselves.

Examples.

(1). $a = 59^\circ\ 28'\ 27''$, $A = 66^\circ\ 7'\ 20''$, $B = 52^\circ\ 50'\ 20''$.

$$b = 48^\circ\ 39'\ 16'', \text{ or } 131^\circ\ 20'\ 44''.$$

Firstly—take the value of b less than a right angle.

$$L \cos \tfrac{1}{2}(a - b) = L \cos\ 5^\circ\ 24'\ 35\tfrac{1}{2}'' = 9{\cdot}9980612.$$
$$L \cos \tfrac{1}{2}(a + b) = L \cos 54^\circ\ 3'\ 51\tfrac{1}{2} = 9{\cdot}7685470.$$
$$L \cot \tfrac{1}{2}(A + B) = L \cot 59^\circ\ 28'\ 50'' = 9{\cdot}7704854.$$

$$\tfrac{1}{2} C = 45^\circ.$$

$$L \cos \tfrac{1}{2}(A + B) = L \cos 59^\circ\ 28'\ 50'' = 9{\cdot}7057190.$$
$$L \cos \tfrac{1}{2}(A - B) = L \cos\ 6^\circ\ 38'\ 30'' = 9{\cdot}9970756.$$
$$L \tan \tfrac{1}{2}(a + b) = L \tan 54^\circ\ 3'\ 51\tfrac{1}{2}'' = 10{\cdot}1397643.$$

$$\tfrac{1}{2} c = 35^\circ\ 11'\ 50\tfrac{3}{4}''.$$

The second value of b is inadmissible (see foot note, Art. 67, Ex. 4), and therefore there is only one solution.

Ans.—

$$b = 48^\circ\ 39'\ 16'', \quad C = 90^\circ, \quad c = 70^\circ\ 23''\ 41\tfrac{1}{2}''.$$

(2). $a = 53^\circ\ 18'\ 20''$, $A = 79^\circ\ 30'\ 45''$, $B = 46^\circ\ 15'\ 15''$.

$$b = 36^\circ\ 5'\ 34\tfrac{3}{4}'', \text{ or } 143^\circ\ 54'\ 25\tfrac{1}{4}''.$$

Firstly—

$$L\cos\tfrac{1}{2}(a-b) = L\cos\ 8^\circ\ 36'\ 22\tfrac{1}{2}'' = 9{\cdot}9950822.$$
$$L\cos\tfrac{1}{2}(a+b) = L\cos 44^\circ\ 41'\ 57\tfrac{1}{4}'' = 9{\cdot}8517528.$$
$$L\cot\tfrac{1}{2}(A+B) = L\cot 62^\circ\ 53'\ \ 0'' = 9{\cdot}7093488.$$
$$\tfrac{1}{2}C = 35^\circ\ 27'\ 47\tfrac{1}{2}''.$$
$$L\cos\tfrac{1}{2}(A+B) = L\cos 62^\circ\ 53'\ \ 0'' = 9{\cdot}6587780.$$
$$L\cos\tfrac{1}{2}(A-B) = L\cos 16^\circ\ 37'\ 45'' = 9{\cdot}9814457.$$
$$L\tan\tfrac{1}{2}(a+b) = L\tan 44^\circ\ 41'\ 57\tfrac{1}{4}'' = 9{\cdot}9954404.$$
$$\tfrac{1}{2}c = 25^\circ\ 12'\ 28\tfrac{1}{2}''.$$

[No ambiguity, as in Ex. 1.]

Ans.—

$$b = 36^\circ\ 5'\ 34\tfrac{3}{4}'' \quad C = 70^\circ\ 55'\ 35'', \quad c = 50^\circ\ 24'\ 57''.$$

(3). $a = 46^\circ\ 45'\ 30''$, $A = 73^\circ\ 11'\ 18''$, $B = 61^\circ\ 18'\ 12''$;

therefore $\quad b = 41^\circ\ 52'\ 34\tfrac{3}{4}''$, or $138^\circ\ 7'\ 25\tfrac{1}{4}'$.

Firstly—

$$L\cos\tfrac{1}{2}(a-b) = L\cos\ 2^\circ\ 26'\ 23\tfrac{1}{4}'' = 9{\cdot}9996061.$$
$$L\cos\tfrac{1}{2}(a+b) = L\cos 44^\circ\ 19'\ \ 6\tfrac{3}{4}'' = 9{\cdot}8545894.$$
$$L\cot\tfrac{1}{2}(A+B) = L\cot 67^\circ\ 14'\ 45'' = 9{\cdot}6226494.$$
$$\tfrac{1}{2}C = 30^\circ\ 21'\ 23\tfrac{1}{4}''.$$
$$L\cos\tfrac{1}{2}(A+B) = L\cos 67^\circ\ 14'\ 45'' = 9{\cdot}5874618.$$
$$L\cos\tfrac{1}{2}(A-B) = L\cos\ 5^\circ\ 56'\ 33'' = 9{\cdot}9976600.$$
$$L\tan\tfrac{1}{2}(a+b) = L\tan 44^\circ\ 19'\ \ 6\tfrac{3}{4}'' = 9{\cdot}9896683.$$
$$\tfrac{1}{2}c = 20^\circ\ 47'\ 32''.$$

[Only one solution.]

Ans.—

$$b = 41^\circ\ 52'\ 34\tfrac{3}{4}'', \quad C = 60^\circ\ 42'\ 46\tfrac{1}{2}'', \quad c = 41^\circ\ 35'\ 4''.$$

(4). $a = 42^\circ\ 15'\ 20'$, $A = 46^\circ\ 30'\ 40''$, $B = 36^\circ\ 20'\ 20''$.

$$b = 33^\circ\ 18'\ 47\tfrac{1}{2}'', \text{ or } 146^\circ\ 41'\ 12\tfrac{1}{2}''.$$

Firstly—

$$L \cos \tfrac{1}{2}(a - b) = L \cos\ 4^\circ\ 28'\ 16\tfrac{1}{4}'' = 9{\cdot}9986763.$$
$$L \cos \tfrac{1}{2}(a + b) = L \cos 37^\circ\ 47'\ 3\tfrac{3}{4}'' = 9{\cdot}8978042.$$
$$L \cot \tfrac{1}{2}(A + B) = L \cot 41^\circ\ 25'\ 30'' = 10{\cdot}0543373.$$
$$\tfrac{1}{2} C = 55^\circ\ 1'\ 37\tfrac{1}{3}''.$$

$$L \cos \tfrac{1}{2}(A + B) = L \cos 41^\circ\ 25'\ 30'' = 9{\cdot}8749585.$$
$$L \cos \tfrac{1}{2}(A - B) = L \cos\ 5^\circ\ 5'\ 10'' = 9{\cdot}9982866.$$
$$L \tan \tfrac{1}{2}(a + b) = L \tan 37^\circ\ 47'\ 3\tfrac{3}{4}'' = 9{\cdot}8894377.$$
$$\tfrac{1}{2} c = 30^\circ\ 16'\ 3''.$$

[Only one solution.]

Ans.—

$$b = 33^\circ\ 18'\ 47\tfrac{1}{2}'', \quad C = 110^\circ\ 3'\ 14\tfrac{2}{3}'', \quad c = 60^\circ\ 32'\ 6''.$$

(5). $a = 34^\circ\ 30'$, $A = 61^\circ\ 29'\ 30''$, $B = 24^\circ\ 30'\ 30''$.

$$b = 15^\circ\ 30'\ 30\tfrac{1}{2}'', \text{ or } 164^\circ\ 29'\ 29\tfrac{1}{2}''.$$

$$L \cos \tfrac{1}{2}(a - b) = L \cos\ 9^\circ\ 29'\ 44\tfrac{3}{4}'' = 9{\cdot}9940081.$$
$$L \cos \tfrac{1}{2}(a + b) = L \cos 25^\circ\ 0'\ 15\tfrac{1}{4}'' = 9{\cdot}9572609.$$
$$L \cot \tfrac{1}{2}(A + B) = L \cot 43^\circ\ 0\ \ 0'' = 10{\cdot}0303441.$$
$$\tfrac{1}{2} C = 49^\circ\ 24'\ 29\tfrac{1}{4}''.$$

$$L \cos \tfrac{1}{2}(A + B) = L \cos 43^\circ\ 0\ \ 0'' = 9.8641275.$$
$$L \cos \tfrac{1}{2}(A - B) = L \cos 18^\circ\ 29'\ 30'' = 9{\cdot}9769777.$$
$$L \tan \tfrac{1}{2}(a + b) = L \tan 25^\circ\ 0'\ 15\tfrac{1}{4}'' = 9{\cdot}6687563.$$
$$\tfrac{1}{2} c = 19^\circ\ 46'\ 56'.$$

[Only one solution.]

Ans.—

$$b = 15^\circ\ 30'\ 30\tfrac{1}{2}'', \quad C = 98^\circ\ 48'\ 58\tfrac{1}{2}'', \quad c = 39^\circ\ 33'\ 52'.$$

(6). $a = 42^\circ\ 15'\ 20''$, $A = 36^\circ\ 20'\ 20''$, $B = 46^\circ\ 30'\ 40''$;

therefore $b = 55^\circ\ 25\ \ 2\tfrac{1}{2}''$, or $124^\circ\ 34'\ 57\tfrac{1}{2}''$.

Firstly—take the value of b less than a right angle.

$$L\cos\tfrac{1}{2}(a-b) = L\cos\ 6^\circ\ 34'\ 51\tfrac{1}{4}'' = 9\cdot9971289.$$
$$L\cos\tfrac{1}{2}(a+b) = L\cos 48^\circ\ 50'\ 11\tfrac{1}{4}'' = 9\cdot8183648.$$
$$L\cot\tfrac{1}{2}(A+B) = L\cot 41^\circ\ 25'\ 30'' = 10\cdot0543373.$$
$$\tfrac{1}{2}C = 59^\circ\ 41'\ 13\tfrac{3}{4}''.$$
$$L\cos\tfrac{1}{2}(A+B) = L\cos 41^\circ\ 25'\ 30'' = 9\cdot8749585.$$
$$L\cos\tfrac{1}{2}(A-B) = L\cos\ 5^\circ\ 5'\ 10'' = 9\cdot9982866.$$
$$L\tan\tfrac{1}{2}(a+b) = L\tan 48^\circ\ 50'\ 11\tfrac{1}{4}'' = 10\cdot0583343.$$
$$\tfrac{1}{2}c = 40^\circ\ 43'\ 43\tfrac{1}{8}''.$$

Secondly—take the value of b greater than a right angle.

$$L\cos\tfrac{1}{2}(a-b) = L\cos 41^\circ\ 9'\ 48\tfrac{3}{4}'' = 9\cdot8766992.$$
$$L\cos\tfrac{1}{2}(a+b) = L\cos 83^\circ\ 25'\ 8\tfrac{3}{4}'' = 9\cdot0592074.$$
$$\tfrac{1}{2}C = 82^\circ\ 20'\ 57\tfrac{1}{2}''.$$
$$L\tan\tfrac{1}{2}(a+b) = L\tan 83^\circ\ 25'\ 8\tfrac{3}{4}'' = 10\cdot9379216.$$
$$\tfrac{1}{2}c = 81^\circ\ 17'\ 13\tfrac{1}{2}''.$$

Ans.—$b = 55^\circ\ 25'\ 2\tfrac{1}{2}''$, $C = 119^\circ\ 22'\ 27\tfrac{1}{2}''$, $c = 81^\circ\ 27'\ 26\tfrac{1}{4}''$,
or $b = 124^\circ\ 34'\ 57\tfrac{1}{2}''$, $C = 164^\circ\ 41'\ 55''$, $c = 162^\circ\ 34'\ 27''$.

(7). $a = 59^\circ\ 28'\ 27''$. $A = 52^\circ\ 50'\ 20''$, $B = 66^\circ\ 7'\ 20''$,
therefore $b = 81^\circ\ 15'\ 15''$, or $98^\circ\ 44'\ 45''$.

Firstly—take the value of b less than a right angle.

$$L\cos\tfrac{1}{2}(a-b) = L\cos 10^\circ\ 53'\ 24'' = 9\cdot9921078.$$
$$L\cos\tfrac{1}{2}(a+b) = L\cos 70^\circ\ 21'\ 51'' = 9\cdot5263918.$$
$$L\cot\tfrac{1}{2}(A+B) = L\cot 59^\circ\ 28'\ 50'' = 9\cdot7704854.$$
$$\tfrac{1}{2}C = 59^\circ\ 51'\ 54''.$$
$$L\cos\tfrac{1}{2}(A+B) = L\cos 59^\circ\ 28'\ 50'' = 9\cdot7057190.$$
$$L\cos\tfrac{1}{2}(A-B) = L\cos\ 6^\circ\ 38'\ 30'' = 9\cdot9970756.$$
$$L\tan\tfrac{1}{2}(a+b) = L\tan 70^\circ\ 21'\ 51'' = 10\cdot4475887.$$
$$\tfrac{1}{2}c = 55^\circ\ 5''\ 25\tfrac{1}{4}''.$$

Secondly—take the value of b greater than a right angle.

$$L\cos\tfrac{1}{2}(a-b) = L\cos 19^\circ\ 38'\ 9'' = 9\cdot9739806.$$
$$L\cos\tfrac{1}{2}(a+b) = L\cos 79^\circ\ 6'\ 36'' = 9\cdot2762872.$$
$$\tfrac{1}{2}C = 71^\circ\ 12'\ 29\tfrac{1}{2}''.$$
$$L\tan\tfrac{1}{2}(a+b) = L\tan 79^\circ\ 6'\ 36'' = 10\cdot7158207.$$
$$\tfrac{1}{2}c = 69^\circ\ 22'\ 43''.$$

Ans.—$b = 81^\circ\ 15'\ 15''$, $C = 119^\circ\ 43'\ 48''$, $c = 110^\circ\ 10'\ 50\tfrac{1}{2}''$;
or $b = 98^\circ\ 44'\ 45''$, $C = 142^\circ\ 24'\ 59''$, $c = 138^\circ\ 45'\ 26''$

70. **Discussion of the Ambiguous Case.**—Before entering upon the general discussion of the ambiguous case of spherical triangles, it will be necessary for the student to be familiar with the analogous ambiguity *in plano*, a complete discussion of which is generally given in works on Plane Trigonometry.

The reasoning is somewhat more complex in the case of spherical triangles, as there are nine distinct cases to be considered, in any of which an ambiguity in the construction of the triangle may present itself. It has been seen in the examples of Arts. 68 and 69, that from the given parts two real solutions were obtained, from which we infer that there can be constructed two distinct triangles satisfying the given conditions.

We shall now examine in detail the variety of cases which may arise when '*two sides and the angle opposite one of them are given*,' leaving the supplemental case, '*having given two angles and the side opposite one of them*,' to be reasoned out in an analogous manner.

71. *Let the given parts be denoted by A, a, b.*

The discussion resolves itself thus into three groups, each containing three separate cases:—

Case I.—When $A = \frac{1}{2}\pi$, and b is equal to, less than, or greater than $\frac{1}{2}\pi$;

Case II.—When $A < \frac{1}{2}\pi$, and b is equal to, less than, or greater than $\frac{1}{2}\pi$;

Case III.—When $A > \frac{1}{2}\pi$, and b is equal to, less than, or greater than $\frac{1}{2}\pi$;

the solution in any case being impossible, evanescent, unique, ambiguous, or indeterminate, according to the value of a.

72. Case I.—$A = \frac{1}{2}\pi$.

For the consideration of the three cases in Art. 71, take any great circle AC (fig. 29) equal to the given side b, and at right angles to the great circle ABX.

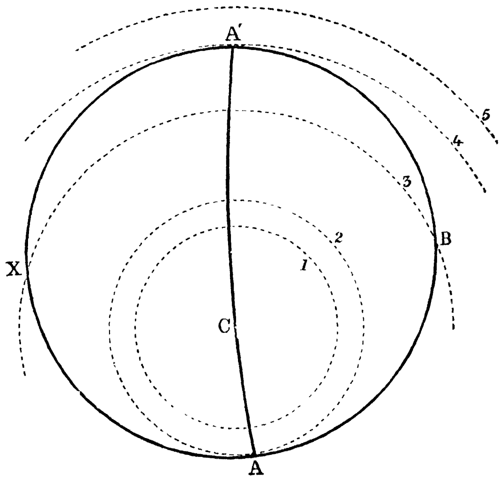

Fig. 29.

Firstly.—Let $b = \frac{1}{2}\pi$; and with C as pole, and a radius equal to a, describe a circle. Now if a be less or greater than a quadrant, this circle being concentric with ABX, the triangle is *impossible.* If a be a quadrant, the circles

are coincident, and the triangle is *indeterminate* with the given parts A, a, b, each quadrants.

Secondly.—Let $b < \frac{1}{2}\pi$. With C as pole (fig. 29), and a radius a, describe a circle. Since AC and $A'C$ are unequal, this circle may be situated, relatively to ABX, in five ways—(1) when the circumference falls wholly within ABX; (2) when it touches ABX at A; (3) when it cuts ABX in the points B and X; (4) when it touches ABX at A'; (5) when it falls without ABX. These different positions are drawn on fig. 29, where it will be seen that the points A and A' are diametrically opposite on ABX; also that B and X are equidistant from the points A and A'.

The following results are therefore geometrically manifest:—

If $a < b$,	the triangle is impossible:	see circle	1.
If $a = b$,	,, ,, evanescent:	,,	2.
If $a > b$ and $< \pi - b$,	two identical solutions:	,,	3.
If $a = \pi - b$,	the triangle is evanescent:	,,	4.
If $a > \pi - b$,	,, ,, impossible:	,,	5.

73. *Thirdly.*—Let $b > \frac{1}{2}\pi$. Let A' be the vertex of the triangle to be constructed with the given parts, and let $A'C = b$. With C as pole, describe a circle of radius a, which, as already explained, may assume any of the five positions (fig. 29), according to the magnitude of a. Hence—

If $a < \pi - b$,	the triangle is impossible:	see circle 1.
If $a = \pi - b$,	" " evanescent:	" 2.
If $a > \pi - b$ and $< b$,	two identical solutions:	" 3.
If $a = b$,	the triangle is evanescent:	" 4.
If $a > b$,	" " impossible:	" 5.

74. Case II.—$A < \frac{1}{2}\pi$.

Draw an arc AC (fig. 30) equal to b, and making the given angle A with the great circle $B_1B_2B_3$. . . . . With C

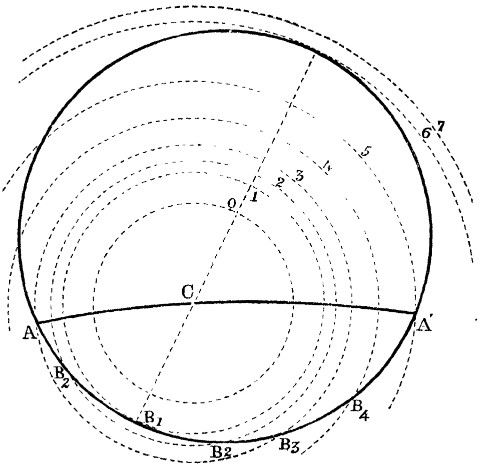

Fig. 30.

as pole, and a radius a, describe a circle. The position of this circle, as in Case I., will depend entirely on the value of the side a; and eight positions, with respect to $B_1B_2B_3$, . . . are shown on fig. 30, viz., circles 0, 1, 2, &c. The circles numbered 0 and 7 respectively have no point in common

with $B_1B_2B_3$ . . ., which we may regard as the base of the triangle under consideration.

We shall discuss the different cases under the heading of this Article—firstly, when $b = \frac{1}{2}\pi$; secondly, when $b < \frac{1}{2}\pi$; thirdly, when $b > \frac{1}{2}\pi$.

Firstly.—Let $b = \frac{1}{2}\pi$. Then C, the vertex of the triangle, is the middle point of the arc AA', and we have—

If $a < A$,	the triangle is		impossible:	see circle 0.
If $a = A$,	,,	,,	unique:	see circle 1.
If $a > A$ and $< b$,	,,	,,	ambiguous:	see circle 2.
If $a = b = \frac{1}{2}\pi$	,,	,,	evanescent:	see circles 3 & 5
If $a > b$,	,,	,,	impossible:	see circles 6 & 7.

Secondly.—Let $b < \frac{1}{2}\pi$. The following results are evident from fig. 30 :—

If $\sin a < \sin b \sin A$, (See circles 0 and 7.)	the triangle is		impossible.
If $\sin a = \sin b \sin A$, (See circle 1.)	,,	,,	unique.
If $\sin a > \sin b \sin A$, and $a < b$, (See circle 2.)	,,	,,	ambiguous.
If $a = b$, (See circle 3.)	,,	,,	unique.
If $a > b$, and $< \pi - b$, (See circle 4.)	,,	,,	unique.
If $a = \pi - b$, (See circle 5.)	,,	,,	evanescent.
If $a > \pi - b$, (See circles 6 and 7.)	,,	,,	impossible.

Thirdly.—Let $b > \frac{1}{2}\pi$. Let $A'C$ (fig. 30) be the given side b, and A' the given angle ($= A$).

Proceeding as before, we have—

If $\sin a < \sin b \sin A$, (See circle 0)	the triangle is impossible.		
If $\sin a = \sin b \sin A$, (See circle 1)	,,	,,	unique.
If $\sin a > \sin b \sin A$, and $a < \pi - b$, (See circle 2.)	,,	,,	ambiguous.
If $a = \pi - b$, (See circle 3.)	,,	,,	unique.
If $a > \pi - b$, and $< b$, (See circle 4.)	,,	,,	unique.
If $a = b$, (See circle 5.)	,,	,,	evanescent.
If $a > b$, (See circles 6 and 7.)	,,	,,	impossible.

75. In the previous Articles, the general cases of ambiguity, when A is equal to, or less than, a quadrant, and a and b any values between 0° and π, have been discussed. The following additional cases are left as exercises for the reader when A is obtuse.

Examples.

1. Given $A > \frac{1}{2}\pi$ and $b < \frac{1}{2}\pi$, the triangle is impossible if $a < b$.

2. In the same case the triangle is unique when a lies between b and $\pi - b$.

3. Given A and b, as in Ex. 1, the triangle is ambiguous when a lies between $\pi - b$ and the greater value of $\sin^{-1}(\sin b \sin A)$.

4. Given A obtuse and b acute, the triangle is impossible if a is greater than the greater value of $\sin^{-1}(\sin b \sin A)$.

5. Given A and b both obtuse, the triangle is impossible when $a < b$.

6 Given A and b both obtuse; find the limiting values of a giving rise to an ambiguity.

Ans. $a > b$, and $< \sin^{-1}(\sin b \sin A)$.

7. What values assigned to the given parts give rise to an *indeterminate* construction?

*8. If a, b, c are each $< \frac{1}{2}\pi$, the greater angle alone *may* exceed $\frac{1}{2}\pi$.

[Let $a > b > c$; then both $\cos b$ and $\cos c > \cos a$, and therefore $> \cos c \cos a$, or $\cos a \cos b$; therefore, &c, Art. 26 (1).]

*9. If a alone $> \frac{1}{2}\pi$, A *must* exceed $\frac{1}{2}\pi$.

[Apply Art. 26 (1).]

*10. If a and b are each $> \frac{1}{2}\pi$, and $c < \frac{1}{2}\pi$; prove that—

(1) the greatest angle (A) must be $> \frac{1}{2}\pi$;

(2) B may be $> \frac{1}{2}\pi$;

(3) C may or may not be $< \frac{1}{2}\pi$.

[Proof as before.]

*11. If $\cos a$, $\cos b$, $\cos c$ are all negative; then $\cos A$, $\cos B$, $\cos C$ are all necessarily negative.

12. In a spherical triangle of the five products,

$\cos a \cos A$, $\cos b \cos B$, $\cos c \cos C$, $\cos a \cos b \cos c - \cos A \cos B \cos C$,

one is negative, the other four being positive.—(COTTERILL.)

[Apply Examples 8, 9, 10, 11.]

76. **The Subsidiary Angle.**—It has been seen that, when two sides and the included angle of a triangle were given, the value of the third side was made to depend on the values ascertained for the base angles. It is likewise evident, in CASES V. and VI. of this Chapter, that the values

* *Educational Times*, vol. xli., reprint, p. 89.

of two of the parts depended on the value or values determined for the remaining part. An error in this latter will, consequently, cause an error in the values of parts found from it.

By means of the subsidiary angle, each of the parts required is determined from the data in question in a manner independent of one another.

(1) In CASE III., Art. 62, c may be found in terms of a, b, C, from a formula adapted to logarithmic computation: thus—

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$
$$= \cos b\,(\cos a + \sin a \tan b \cos C).$$

Let $$\tan \theta = \tan b \cos C. \tag{1}$$

Hence

$$\cos c = \cos b\,(\cos a + \sin a \tan \theta) = \cos b \cos (a - \theta) \sec \theta;$$

or $$\frac{\cos b}{\cos c} = \frac{\cos \theta}{\cos (a - \theta)}. \tag{2}$$

The subsidiary angle θ is found from (1), and c can therefore be calculated from (2).

NOTE.—It is evident, from (2), that θ and $a - \theta$ are the segments of a, made by the perpendicular to it from the opposite angle A. (Cf. Chap. IV., Ex. 11.) The solution of the triangle is thus shown to be equivalent to the solutions of these right-angled triangles.

(2) In CASE IV., Art 67, C may be found directly from

the given parts A, B, c, in a form adapted to logarithmic computation: thus—

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

$$= \cos B\,(-\cos A + \sin A \tan B \cos c).$$

Let $$\cot\theta = \tan B \cos c.$$

Hence

$$\cos C = \cos B\,(-\cos A + \sin A \cot\theta) = \cos B \sin(A-\theta)\operatorname{cosec}\theta;$$

or $$\frac{\cos B}{\cos C} = \frac{\sin\theta}{\sin(A-\theta)};\ \text{therefore, \&c.}$$

It is obvious that the subsidiary angle θ is in this case a segment of the angle A made by the perpendicular from A on the opposite side. (Cf. Chap. IV., Ex. 17.)

Examples.

1. Having given a, b, A, adapt the formula

$$\cot a \sin b = \cot A \sin C + \cos b \cos C$$

to logarithmic computation for the angle C.

Ans. $\tan a : \tan b = \cos\theta : \cos(C-\theta.)$

2. What is the geometrical interpretation of θ in Ex. 1?

[Cf. Art. 44 (2).]

3. Given, as before, a, b, A; show how to find c in a logarithmic form.

Ans. $\cos a : \cos b = \cos(c-\theta) : \cos\theta$, where $\tan\theta = \tan b \cos A$.

[Cf. Art. 26 (1).]

4. Prove the result given in Ex. 3 geometrically.

5. Having given A, B, a; construct geometrically the subsidiary angles required—(1) to calculate C; (2) to calculate c.

Ans. Draw a perpendicular from C to c, and take—(1) the segment of C adjacent to a; (2) the segment of c adjacent to a.

6. Having given $a = 84° 30'$, $b = 56° 46'$, $C = 73° 52'$; calculate c by means of a subsidiary angle.

By aid of Art. 76, we have

$$L \tan\theta = L \tan b + L \cos C - 10$$

$$= 9{\cdot}6274648;$$

therefore $$\theta = 22° 58' 54''.$$

Again, $$L \cos c = L \cos b + L \cos(a - \theta) - L \cos\theta$$

$$= 9{\cdot}4531418;$$

therefore $$c = 73° 30' 28''.$$

7. Having given $a = 71° 35' 16''$, $b = 40° 33' 12''$, $C = 103° 57' 20\tfrac{1}{4}''$; calculate c by means of a subsidiary angle.

As in Ex. 6,

$$L \tan \text{supplement of } \theta = 9{\cdot}3146412; \text{ therefore } \theta = 168° 20' 23''.$$

$$L \cos C = 8{\cdot}9600566; \text{ therefore } c = 84° 46'.$$

8. Having given $A = 54° 40'$, $B = 75° 20'$, $c = 40° 0' 4''$; calculate C by means of a subsidiary angle

By Art 76,

$$L \cot\theta = L \tan B + L \cos c - 10;$$

therefore $$\theta = 18° 51' 48''.$$

Again, $$L \cos c = L \cos B + L \sin(A - \theta) - L \sin\theta$$

$$= 9{\cdot}6609932;$$

therefore $$C = 62° 44'.$$

9. Having given $A = 56° 58'$, $B = 39° 26'$, $c = 68° 48'$; calculate C by means of a subsidiary angle.

Ans. $\theta = 73° 26' 16''$, $C = 103° 12' 25''$.

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